

Lecture Outline 2: Euclidean Spaces, Linear Independence

This lecture note is based on Chapter 10 of *Mathematics for Economists* by Simon and Blume.

1. Points, Lines and Planes: Basic objects of Euclidean spaces
2. Vectors

- Displacement \vec{pq}

- Addition and subtraction

- Scalar multiplication

- Length and distance

- * $\|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2}$

- * $\|rv\| = |r|\|v\|$ (Q: can you prove it?)

- Inner Product $u \cdot v$

- * $\|u - v\| = \sqrt{(u - v) \cdot (u - v)}$

- * $u \cdot v = \|u\| \|v\| \cos \theta$ (Again: let's prove it?)

- * Orthogonal vectors

- * Triangle inequality $\|u + v\| \leq \|u\| + \|v\|$ (Q: when does the equality hold? Also see exercise 10.22)

- * $|\|u\| - \|v\|| \leq \|u - v\|$

Exercise For each of the following vectors, find a vector of length 1 which points in the same direction: a) (3,4), b) (6,0), c) (1, 1, 1), d) (-1,2,-3).

Exercise Prove that $\|u - v\|^2 = \|u\|^2 - 2u \cdot v + \|v\|^2$

3. Lines and Planes

- Parametric representation

- * line: $x(t) = x_0 + tv$ (Q: is this representation unique?), or linear combinations of two points

- * plane: $x = p + sv + tw$, or linear combination of three points

- Nonparametric representation: Find normal vector (Q: how do you switch between a nonparametric equation and a parametric one?)

- Hyperplane: Hyperplane in R^n has dimension $n - 1$

Exercise Show that the midpoint of $L(x,y)$ occurs where $t = \frac{1}{2}$. In other words, if

$$z = \frac{1}{2}x + \frac{1}{2}y, \text{ show that } \|x - z\| = \|y - z\|$$

Exercise Does the point $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ lie on the plane $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$?

Exercise Derive parametric and nonparametric equations for the lines which pass through (1,2) and (3,6).

4. Linear Independence

- Definition

- * the set spanned by v_1 and v_2 : $L[v_1, v_2] \equiv \{r_1v_1 + r_2v_2 : r_1 \in R \text{ and } r_2 \in R\}$
- * v_1 and v_2 are linearly independent if $r_1v_1 + r_2v_2 = 0 \Rightarrow r_1 = r_2 = 0$

● How to check linear independence?

- * check the solution of the corresponding linear system
- * check the determinant
- * If $k > n$, any set of k vectors in R^n is linearly dependent

Exercise Are $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ linearly independent?

5. Spanning Sets

- Q: can different sets of vectors span the same space (see example 11.6)?
- Q: how many vectors does it take to span R^n ?

Exercise Do $\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$ span R^3 ? If not, find an example of a vector not in their span. (Hint: go to pages 192 for the determinant.)

6. Basis and Dimension in R^n

- Basis: any set of linearly independent vectors that span R^n
- Canonical basis
- Every basis of R^n contains n vectors

Exercise In Example 11.6, which sets of vectors are basis for R^2 ?