SECTION 1

Affine Sets

Throughout this book, R denotes the real number system, and R^n is the usual vector space of real n-tuples $x = (\xi_1, \ldots, \xi_n)$. Everything takes place in R^n unless otherwise specified. The inner product of two vectors x and x^* in R^n is expressed by

$$\langle x, x^* \rangle = \xi_1 \xi_1^* + \cdots + \xi_n \xi_n^*$$

The same symbol A is used to denote an $m \times n$ real matrix A and the corresponding linear transformation $x \to Ax$ from R^n to R^m. The transpose matrix and the corresponding adjoint linear transformation from R^m to R^n are denoted by A^*, so that one has the identity

$$\langle Ax, y^* \rangle = \langle x, A^* y^* \rangle.$$

(In a symbol denoting a vector, * has no operational significance; all vectors are to be regarded as column vectors for purposes of matrix multiplication. Vector symbols involving * are used from time to time merely to bring out the familiar duality between vectors considered as points and vectors considered as the coefficient n-tuples of linear functions.) The end of a proof is signalled by ‖.

If x and y are different points in R^n, the set of points of the form

$$(1 - \lambda)x + \lambda y = x + \lambda(y - x), \qquad \lambda \in R,$$

is called the *line through x and y*. A subset M of R^n is called an *affine set* if $(1 - \lambda)x + \lambda y \in M$ for every $x \in M$, $y \in M$ and $\lambda \in R$. (Synonyms for "affine set" used by other authors are "affine manifold," "affine variety," "linear variety" or "flat.")

The empty set $\emptyset$ and the space R^n itself are extreme examples of affine sets. Also covered by the definition is the case where M consists of a solitary point. In general, an affine set has to contain, along with any two different points, the entire line through those points. The intuitive picture is that of an endless uncurved structure, like a line or a plane in space.

The formal geometry of affine sets may be developed from the theorems of linear algebra about subspaces of R^n. The exact correspondence between affine sets and subspaces is described in the two theorems which follow.