# Sequences and Convergence in Metric Spaces

**Definition:** A **sequence** in a set $X$ (a sequence of elements of $X$) is a function $s : \mathbb{N} \to X$. We usually denote $s(n)$ by $s_n$, called *the $n$-th term of $s$*, and write $\{s_n\}$ for the sequence, or $\{s_1, s_2, \ldots\}$.

See the nice introductory paragraphs about sequences on page 23 of de la Fuente.

**Remark:** Let $V$ be a vector space. The set of sequences in $V$ is a vector space under the natural component-wise definitions of vector addition and scalar multiplication:

$$\{x_1, x_2, \ldots\} + \{y_1, y_2, \ldots\} := \{x_1 + y_1, x_2 + y_2, \ldots\} \quad \text{and} \quad \alpha\{x_1, x_2, \ldots\} := \{\alpha x_1, \alpha x_2, \ldots\}.$$

By analogy with $\mathbb{R}^n$, we use the notation $\mathbb{R}^\infty$ to denote the set of sequences of real numbers, and we use the notation $X^\infty$ to denote the set of sequences in a set $X$. (But $\infty$ is *not* an element of $\mathbb{N}$, a natural number, so this notation is indeed simply an analogy.)

When we defined $\mathbb{R}^n$ as a vector space, we defined vector addition and scalar multiplication in $\mathbb{R}^n$ component-wise, from the addition and multiplication of the real-number components of the $n$-tuples in $\mathbb{R}^n$, just as we've done in the remark above for $V^\infty$. Unlike $\mathbb{R}^n$, however, the vector space $V^\infty$ is not finite-dimensional, even if $V$ *is* finite-dimensional. For example, no finite subset forms a basis of the set $\mathbb{R}^\infty$ of sequences of real numbers (*i.e.*, sequences in $\mathbb{R}$).

Because $\mathbb{R}^\infty$ is a vector space, we could potentially define a norm on it. But the norms we defined on $\mathbb{R}^n$ don't generalize in a straightforward way to $\mathbb{R}^\infty$. For example, you should be able to easily provide an example of a sequence for which neither $\max\{|x_1|, |x_2|, \ldots\}$ nor $\sum_{n=1}^{\infty} |x_n|$ is a real number. This is one symptom of the fact that the set of *all* sequences in a space generally doesn't have nice properties. But we're often interested only in sequences that *do* have nice properties — for example, the set of all *bounded* sequences.

**Definition:** A sequence $\{x_n\}$ of real numbers is **bounded** if there is a number $M \in \mathbb{R}$ for which every term $x_n$ satisfies $|x_n| \leqq M$. More generally, a sequence $\{x_n\}$ in a normed vector space is **bounded** if there is a number $M \in \mathbb{R}$ for which every term $x_n$ satisfies $||x_n|| \leqq M$.

**Remark:** We use the notation $\ell^\infty$ for the set of all bounded real sequences, equipped with the norm $||\{x_n\}||_\infty := \sup\{|x_1|, |x_2|, \ldots\}$. Note that this is a subset of $\mathbb{R}^\infty$. Note too that we needed to change *max* to *sup* in the definition of $||\{x_n\}||_\infty$ in order that the norm be well-defined: the sequence $x_n = 1 - (1/n)$, for example, is in $\ell^\infty$ (it's bounded), and $\sup\{|x_n| \mid n \in \mathbb{N}\} = 1$, but $\max\{|x_n| \mid n \in \mathbb{N}\}$ is not defined.

**Exercise:** Verify that $\ell^\infty$ is a vector subspace of $\mathbb{R}^\infty$ and that $||\{x_n\}||_\infty$ is indeed a norm. Therefore $\ell^\infty$ is a normed vector space.

We can easily convert our definition of bounded sequences in a normed vector space into a definition of bounded sets and bounded functions. And by replacing the norm in the definition with the distance function in a metric space, we can extend these definitions from normed vector spaces to general metric spaces.

**Definition:** A subset $S$ of a metric space $(X, d)$ is **bounded** if

$$\exists \bar{x} \in X, M \in \mathbb{R} : \forall x \in S : d(x, \bar{x}) \leqq M.$$

A function $f : D \to (X, d)$ is **bounded** if its image $f(D)$ is a bounded set.

Since a sequence in a metric space $(X, d)$ is a function from $\mathbb{N}$ into $X$, the definition of a bounded function that we've just given yields the result that a sequence $\{x_n\}$ in a metric space $(X, d)$ is bounded if and only if

$$\exists \bar{x} \in X, M \in \mathbb{R} : \forall n \in \mathbb{N} : d(x_n, \bar{x}) \leqq M,$$

so that the definition we gave earlier for a bounded sequence of real numbers is simply a special case of this more general definition.

**Remark:** Every element of $C[0, 1]$ is a bounded function.

**Question:** Is $C[0, 1]$ a bounded set, for example under the max norm $||f||_\infty$ ?

**Exercise:** Let $S$ be the set of all real sequences that have only a finite number of non-zero terms — *i.e.*, $S = \{\{x_n\} \in \mathbb{R}^\infty \mid x_n \neq 0 \text{ for a finite set } A \subseteq \mathbb{N}\}$ . Determine whether $S$ is a vector subspace of $\ell^\infty$ . If it is, provide a proof; if it isn't, show why not.

Another important set of sequences is the set of *convergent* sequences, which we study next.

## Convergence of Sequences

**Definition:** A sequence $\{x_n\}$ of real numbers **converges** to $\bar{x} \in \mathbb{R}$ if

$$\forall \epsilon > 0 : \exists \bar{n} \in \mathbb{N} : n > \bar{n} \Rightarrow |x_n - \bar{x}| < \epsilon.$$

**Example 1:** The sequence $x_n = \dfrac{1}{n}$ converges to 0.

**Example 2:** The sequence $x_n = (-1)^n$ does not converge.

**Example 3:** The sequence

$$x_n = \begin{cases} 1, & \text{if } n \text{ is a square, } \textit{i.e.} \text{ if } n \in \{1, 4, 9, 16, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

does not converge, despite the fact that it has ever longer and longer strings of terms that are zero.

If we replace $|x_n - \bar{x}|$ in this definition with the distance notation $d(x_n, \bar{x})$, then the definition applies to sequences in any metric space:

**Definition:** Let $(X, d)$ be a metric space. A sequence $\{x_n\}$ in $X$ **converges** to $\bar{x} \in X$ if

$$\forall \epsilon > 0 : \exists \bar{n} \in \mathbb{N} : n > \bar{n} \Rightarrow d(x_n, \bar{x}) < \epsilon.$$

We say that $\bar{x}$ is the limit of $\{x_n\}$, and we write $\lim\{x_n\} = \bar{x}, x_n \to \bar{x}$, and $\{x_n\} \to \bar{x}$ .

**Remark:** A convergent sequence in a metric space is bounded. Therefore the set of convergent real sequences is a subset of $\ell^\infty$ .

**Exercise:** Provide a proof that the set of convergent real sequences is a vector subspace of $\ell^\infty$. This is not quite as trivial as it might at first appear. As usual, you have to show that the set in question (the set of convergent sequences) is closed under vector addition and scalar multiplication. So you have to show that if two sequences $\{x_n\}$ and $\{y_n\}$ both converge, then the sequences $\{x_n\} + \{y_n\}$ and $\alpha\{x_n\}$ both converge.

**Example:** For each $n \in \mathbb{N}$, let $a_n = 1 - \frac{1}{n}$ and define $F_n : [0, 1] \to \mathbb{R}$ by

$$F_n(x) = \begin{cases} \dfrac{1}{a_n} x & , \text{ if } x \leqq a_n \\ 1 & , \text{ if } x \geqq a_n. \end{cases}$$

Then $F_n \to F$, where $F(x) = x$. What is the value of $||F_n - F||$ ?

Note that $F_n$ in this example is the cdf of the uniform distribution on the interval $[0, a_n]$, and $\{F_n\}$ converges to the cdf of the uniform distribution on $[0, 1]$.

**Definition:** Let $(X, d)$ be a metric space, let $\bar{x} \in X$, and let $r \in \mathbb{R}_{++}$.
(1) The open ball about $\bar{x}$ of radius $r$ is the set $B(\bar{x}, r) := \{x \in X \mid d(x, \bar{x}) < r\}$.
(2) The closed ball about $\bar{x}$ of radius $r$ is the set $\bar{B}(\bar{x}, r) := \{x \in X \mid d(x, \bar{x}) \leqq r\}$.

**Remark:** Let $(X, d)$ be a metric space. A subset $S \subseteq X$ is bounded if and only if it is contained in an open ball — and equivalently, if and only if it is contained in a closed ball.

**Remark:** Let $(X, d)$ be a metric space. A sequence $\{x_n\}$ in $X$ converges to $\bar{x}$ if and only if for every $\epsilon > 0$, "$x_n$ is eventually in $B(\bar{x}, \epsilon)$" — *i.e.*, $\exists \bar{n} \in \mathbb{N} : n > \bar{n} \Rightarrow x_n \in B(\bar{x}, \epsilon)$.

## Subsequences and Cluster Points

**Definition:** Let $f : X \to Y$ be a function and let $A$ be a subset of $X$. The **restriction** of $f$ to $A$, denoted $f|_A$, is the function $f|_A : A \to Y$ defined by

$$\forall x \in A : f|_A(x) = f(x).$$

**Definition:** Let $\{x_n\}$ be a sequence in $X$ — *i.e.*, $x : \mathbb{N} \to X$. A **subsequence** of $\{x_n\}$ denoted $\{x_{n_k}\}$, is the restriction of the function $x(\cdot)$ to an infinite subset of $\mathbb{N}$.

**Remark:** A subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ in $X$ is also a sequence in $X$.

**Example 1:** Let $x_n = n$ and $n_k = k^2$. Then $x_n = \{1, 2, 3, 4, \ldots\}$ and $\{x_{n_k}\} = \{1, 4, 9, 16, \ldots\}$.

**Example 2:** Let $x_n = (-1)^n$ and $n_k = 2k$. Then $\{x_n\} = \{-1, 1, -1, 1, \ldots\}$ and $\{x_{n_k}\} = \{1, 1, 1, \ldots\}$. Or let $n_m = 2m + 1$; then $\{x_{n_m}\} = \{-1, -1, -1, \ldots\}$. Or let $n_k = k^2$; then $\{x_{n_k}\} = \{-1, 1, -1, 1, \ldots\}$.

**Example 3:** Let $x_n = (-1)^n \dfrac{n-1}{n}$ and $n_k = 2k$. Then $\{x_n\} = \{0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, \ldots\}$ and $\{x_{n_k}\} = \{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots\}$. Or let $n_k = 2k + 1$; then $\{x_{n_k}\} = \{-\frac{2}{3}, -\frac{4}{5}, -\frac{6}{7}, \ldots\}$.

**Definition:** Let $\{x_n\}$ be a sequence in a metric space $(X, d)$. A **cluster point** of $\{x_n\}$ is an element $\bar{x} \in X$ such that every open ball around $\bar{x}$ contains an infinite number of terms of $\{x_n\}$ — *i.e.*, such that for every $\epsilon > 0$, the set $\{n \in \mathbb{N} \mid x_n \in B(\bar{x}, \epsilon\}$ is an infinite set. We also say that for every $\epsilon > 0$, "$x_n$ is frequently in $B(\bar{x}, \epsilon\}$."

**Remark:** If $\{x_n\}$ converges to $\bar{x}$, then $\bar{x}$ is its only cluster point. Therefore, if $\{x_n\}$ has more than one cluster point, it doesn't converge.

**Remark:** If $\{x_n\}$ converges to $\bar{x}$, then every subsequence of $\{x_n\}$ converges to $\bar{x}$. Therefore, if $\{x_n\}$ has a subsequence that doesn't converge, then $\{x_n\}$ doesn't converge.

**Remark:** $\bar{x}$ is a cluster point of $\{x_n\}$ if and only if there is a subsequence that converges to $\bar{x}$.

**Examples:** In Example 1, $\{x_n\}$ does not converge and in fact has no cluster points, so it has no convergent subsequences. In Example 2, $\{x_n\}$ has exactly two cluster points, 1 and -1, so the sequence doesn't converge; we exhibited a subsequence that converges to 1, a subsequence that converges to -1, and a subsequence that doesn't converge. In Example 3, $\{x_n\}$ has the same two cluster points, 1 and -1, and for each one we exhibited a subsequence that converges to it.

Here's a proof that the sum of two convergent sequences in a normed vector space is a convergent sequence:

**Proposition:** If $\{x_n\}$ and $\{y_n\}$ are sequences in a normed vector space $(V, ||\cdot||)$ and if $\{x_n\} \to \bar{x}$ and $\{y_n\} \to \bar{y}$, then $\{x_n\} + \{y_n\} \to \bar{x} + \bar{y}$ .

**Proof:** Let $\epsilon > 0$. Because $\{x_n\} \to \bar{x}$ and $\{y_n\} \to \bar{y}$, there are $\bar{n}_x$ and $\bar{n}_y$ such that

$$n > \bar{n}_x \Rightarrow ||x_n - \bar{x}|| < \epsilon/2 \quad \text{and} \quad n > \bar{n}_y \Rightarrow ||y_n - \bar{y}|| < \epsilon/2.$$

Let $\bar{n} = \max\{\bar{n}_x, \bar{n}_y\}$; then

$$n > \bar{n} \Rightarrow ||x_n - \bar{x}|| + ||y_n - \bar{y}|| < \epsilon/2 + \epsilon/2 = \epsilon.$$

But we have

$$\begin{aligned} ||(x_n + y_n) - (\bar{x} + \bar{y})|| &= ||(x_n - \bar{x}) + (y_n - \bar{y})|| \\ &\leqq ||x_n - \bar{x}|| + ||y_n - \bar{y}||, \text{ by the Triangle Inequality.} \end{aligned}$$

Therefore

$$n > \bar{n} \Rightarrow ||(x_n + y_n) - (\bar{x} + \bar{y})|| < \epsilon,$$

and since $x_n + y_n$ is the $n^{th}$ term of the sequence $\{x_n\} + \{y_n\}$, this concludes the proof. □