

Open and Closed Sets

Definition: A subset S of a metric space (X, d) is **open** if it contains an open ball about each of its points — *i.e.*, if

$$\forall x \in S : \exists \epsilon > 0 : B(x, \epsilon) \subseteq S. \quad (1)$$

Theorem:

(a) \emptyset and X are open sets.

(b) If S_1, S_2, \dots, S_n are open sets, then $\bigcap_{i=1}^n S_i$ is an open set.

(c) Let A be an arbitrary set. If S_α is an open set for each $\alpha \in A$, then $\bigcup_{\alpha \in A} S_\alpha$ is an open set. In other words, the union of any collection of open sets is open. [Note that A need not be finite, or even countable. See Examples 4 and 5 below.]

Proof:

(a) \emptyset is open because the condition (1) is vacuously satisfied: there is no $x \in \emptyset$. X is open because any ball is by definition a subset of X .

(b) Let S_i be an open set for $i = 1, \dots, n$, and let $x \in \bigcap_{i=1}^n S_i$. We must find an $\epsilon > 0$ for which $B(x, \epsilon) \subseteq \bigcap_{i=1}^n S_i$. For each i there is an ϵ_i such that $B(x, \epsilon_i) \subseteq S_i$, because each S_i is open. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. Then $0 < \epsilon \leq \epsilon_i$ for each i ; therefore $B(x, \epsilon) \subseteq B(x, \epsilon_i) \subseteq S_i$ for each i — *i.e.*, $B(x, \epsilon) \subseteq \bigcap_{i=1}^n S_i$.

(c) Let $x \in \bigcup_{\alpha \in A} S_\alpha$; we must find an $\epsilon > 0$ for which $B(x, \epsilon) \subseteq \bigcup_{\alpha \in A} S_\alpha$. Since $x \in \bigcup_{\alpha \in A} S_\alpha$, we have $x \in S_{\bar{\alpha}}$ for some $\bar{\alpha}$. Since $S_{\bar{\alpha}}$ is open, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S_{\bar{\alpha}} \subseteq \bigcup_{\alpha \in A} S_\alpha$. \square

Definition: A subset S of a metric space (X, d) is **closed** if it is the complement of an open set.

Theorem:

(a') \emptyset and X are closed sets.

(b') If S_1, S_2, \dots, S_n are closed sets, then $\bigcup_{i=1}^n S_i$ is a closed set.

(c') Let A be an arbitrary set. If S_α is a closed set for each $\alpha \in A$, then $\bigcap_{\alpha \in A} S_\alpha$ is a closed set. In other words, the intersection of any collection of closed sets is closed.

Proof:

(a') follows directly from (a).

(b') and (c') follow from (c) and (b) by De Morgan's Laws. \square

Exercise: Use De Morgan's Laws to establish (b') and (c').

Example 1: For each $n \in \mathbb{N}$, let S_n be the open set $(-\frac{1}{n}, \frac{1}{n}) \subseteq \mathbb{R}$. Then $\bigcap_{n=1}^{\infty} S_n = \{0\}$, which is not open. This is a counterexample which shows that (b) would not be valid if it allowed the collection to be infinite.

Example 2: For each $n \in \mathbb{N}$, let S_n be the closed set $(\frac{1}{n}, \frac{n-1}{n}) \subseteq \mathbb{R}$. Then $\bigcup_{n=1}^{\infty} S_n = (0, 1)$, which is not closed. This is a counterexample which shows that (b') would not be valid if it allowed the collection to be infinite.

Example 3: \mathbb{R}_{++}^n is open. This can be shown directly, by finding an appropriate $\epsilon > 0$ for each $x \in \mathbb{R}$. Alternatively, one could show that for each $i = 1, \dots, n$ the set $S_i = \{x \in \mathbb{R}^n \mid x_i > 0\}$ is open, and then invoke (b') for the set $\mathbb{R}_{++}^n = \bigcap_{i=1}^n S_i$.

Example 4: The union of all open subsets of \mathbb{R}_+^n is an open set, according to (c). Note that this set is \mathbb{R}_{++}^n . This is therefore a third way to show that \mathbb{R}_{++}^n is an open set.

Example 5: Generalizing Example 4, let G be any subset of (X, d) and let S be the union of all open subsets of G . According to (c), S is an open set. S is clearly the “largest” open subset of G , in the sense that (i) S is itself a subset of G , and (ii) every open subset of G is a subset of S — *i.e.* S contains every open subset of G . We call this set the **interior** of G , denoted $int G$.

Example 6: Doing the same thing for closed sets, let G be any subset of (X, d) and let S be the intersection of all closed sets that contain G . According to (c'), S is a closed set. It is the “smallest” closed set containing G as a subset, in the sense that (i) S is itself a closed set containing G , and (ii) every closed set containing G as a subset also contains S as a subset. We call S the **closure** of G , denoted $cl G$.

Note that for any set G , $int G \subseteq G \subseteq cl G$.

Example 7: Let $u : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}$ be defined by $u(x_1, x_2) = x_1x_2$, and let $S = \{x \in \mathbb{R}_{++}^2 \mid u(\mathbf{x}) < \xi\}$ for some $\xi \in \mathbb{R}_{++}$. Note that $S = u^{-1}((-\infty, \xi))$, the inverse image under u of the open interval $(-\infty, \xi)$.

We will show that S is an open set.

Proof: Let $\bar{x} \in S$; we must find an $\epsilon > 0$ such that $B(\bar{x}, \epsilon) \subseteq S$ — *i.e.*, such that

$$x \in B(\bar{x}, \epsilon) \Rightarrow x_1x_2 < \xi. \quad (2)$$

Let's make the arithmetic easier here by exploiting the fact that $\|z\|_\infty \leq \|z\|_2$ for all $z \in \mathbb{R}^2$, so that $B_2(\bar{x}, \epsilon) \subseteq B_\infty(\bar{x}, \epsilon)$ for all $\bar{x} \in \mathbb{R}_{++}^2$ and all $\epsilon > 0$ — *i.e.*,

$$x \in B_2(\bar{x}, \epsilon) \Rightarrow x \in B_\infty(\bar{x}, \epsilon).$$

Therefore, if we can find an $\epsilon > 0$ such that

$$x \in B_\infty(\bar{x}, \epsilon) \Rightarrow x_1x_2 < \xi, \quad (3)$$

that will be sufficient to establish (1).

We can further simplify by noting that

$$x \in B_\infty(\bar{x}, \epsilon) \Rightarrow [x_1 < \bar{x}_1 + \epsilon \ \& \ x_2 < \bar{x}_2 + \epsilon],$$

so now if we can find an $\epsilon > 0$ such that

$$[x_1 < \bar{x}_1 + \epsilon \ \& \ x_2 < \bar{x}_2 + \epsilon] \Rightarrow x_1x_2 < \xi \quad (4)$$

then that will be sufficient to establish (2), and therefore (1). Since all x_1 and x_2 are positive, we always have $x_1x_2 < (\bar{x}_1 + \epsilon)(\bar{x}_2 + \epsilon)$, so if we can find an $\epsilon > 0$ such that

$$(\bar{x}_1 + \epsilon)(\bar{x}_2 + \epsilon) < \xi \quad (5)$$

then that will be sufficient to establish (3), and therefore, in turn, (2) and (1).

In order to solve the inequality (4), we solve for ϵ in the equation

$$(\bar{x}_1 + \epsilon)(\bar{x}_2 + \epsilon) = \xi. \quad (6)$$

Rewriting the equation as

$$\epsilon^2 + (\bar{x}_1 + \bar{x}_2)\epsilon + (\bar{x}_1\bar{x}_2) - \xi = 0 \quad (7)$$

and applying the quadratic formula, we obtain

$$\epsilon = \frac{1}{2} \left[-(\bar{x}_1 + \bar{x}_2) \pm \sqrt{(\bar{x}_1 + \bar{x}_2)^2 + 4\xi - 4\bar{x}_1\bar{x}_2} \right]. \quad (8)$$

Since $\bar{x}_1\bar{x}_2 < \xi$, the positive root yields a positive value of ϵ — call this $\hat{\epsilon}$ — a value that satisfies (5). Therefore any smaller value of ϵ will satisfy (4), and will therefore in turn satisfy (3), and therefore (2), and therefore (1). \square

Example 7 is an important example — not the specific utility function $u(x_1, x_2) = x_1x_2$, but the fact that the set S we defined (the inverse image under u of an open set in \mathbb{R}) was itself an open set in the domain of u . In fact, it's a theorem we'll do later that a function $f : (X, d) \rightarrow (Y, d')$ is continuous if and only if $f^{-1}(A)$ is open in X for every open set $A \subseteq Y$. This has important implications for utility theory and other applications in economics.

Definition: A **limit point** of a set S in a metric space (X, d) is an element $\bar{x} \in X$ for which there is a sequence in S that converges to \bar{x} .

Theorem: A set is closed if and only if it contains all its limit points.

Proof:

(\Rightarrow): Let S be a closed set, and let $\{x_n\}$ be a sequence in S (*i.e.*, $\forall n \in \mathbb{N} : x_n \in S$) that converges to $\bar{x} \in X$. We must show that $\bar{x} \in S$. Suppose not — *i.e.*, $\bar{x} \in S^c$. Since S^c is open, there is an $\hat{\epsilon} > 0$ for which $B(\bar{x}, \hat{\epsilon}) \subseteq S^c$. Since $\{x_n\} \rightarrow \bar{x}$, let \bar{n} be such that $n > \bar{n} \Rightarrow d(x_n, \bar{x}) < \hat{\epsilon} > 0$. Then for $n > \bar{n}$ we have both $x_n \in S$ and $x_n \in B(\bar{x}, \hat{\epsilon}) \subseteq S^c$, a contradiction.

(\Leftarrow): Suppose S is not closed. We must show that S does not contain all its limit points. Since S is not closed, S^c is not open. Therefore there is at least one element \bar{x} of S^c such that every ball $B(\bar{x}, \epsilon)$ contains at least one element of $(S^c)^c = S$. For every $n \in \mathbb{N}$, let $x_n \in B(\bar{x}, \frac{1}{n}) \cap S$. Then we have a sequence $\{x_n\}$ in S which converges to $\bar{x} \notin S$ — *i.e.*, \bar{x} is a limit point of S but is not in S , so S does not contain all its limit points. □