Correspondences

Functions are single-valued. For a function $f : X \to Y$, every $x \in X$ is mapped to one *and only one* point $y \in Y$, the point $y = f(x)$. Thus, for example, when a consumer with a strictly quasiconcave utility function behaves according to the Utility Maximization Hypothesis, we can summarize his market behavior by his demand function: the bundle he chooses is a (single-valued) function of the price-list he faces.

But suppose the consumer's utility function is not strictly quasiconcave; for example, suppose it is $u(x_1, x_2) = ax_1 + x_2$. Whenever the price-list satisfies $p_1 = ap_2$, this consumer is indifferent among *all* the bundles on the indifference curve $\{\mathbf{x} \in \mathbb{R}^2_+ \mid ax_1 + x_2 = a\mathring{x}_1 + \mathring{x}_2\}$. He will of course choose a single bundle, but we can't say which one it will be. We can say only that it will be one of the many utility-maximizing bundles.

This situation, in which we need to analyze behavior that does not manifest itself in uniquely determined actions, is extremely common in economics and game theory. And of course, if individuals' behavior is not single-valued, then aggregate behavior won't be single-valued either. In the demand theory example above, the market demand function will not be single-valued. We need a new analytical tool, the *multivalued function* or *correspondence*.

Definition: A *correspondence* f from a set X to a set Y, denoted $f : X \twoheadrightarrow Y$, is a function from X to the set 2^Y of all subsets of Y. Correspondences are also called multivalued functions or set-valued functions.

Remark: A correspondence $f : X \twoheadrightarrow Y$ is thus a set-valued function from X to Y — for every $x \in X$, $f(x)$ is a subset of Y.

Example 1: $f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ is defined by $f(x) = \{\sqrt{x}, -\sqrt{x}\}$.

Example 2: $f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} [0, \frac{1}{x}] & \text{if } x > 0 \\ \{0\} & \text{if } x = 0. \end{cases}$$

Example 3: $f : \mathbb{R}_+ \longrightarrow\!\!\!\!\rightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} [x, \frac{1}{2}] & \text{if } 0 \leqq x \leqq \frac{1}{2} \\ \{\frac{1}{4}, \frac{1}{2}\} \cup [\frac{3}{4}, 1] & \text{if } x = \frac{1}{2} \\ \{\frac{1}{4}\} \cup [\frac{3}{4}, 1] & \text{if } x \geqq \frac{1}{2} \end{cases}$$

Definition: The *graph* of a correspondence $f : X \longrightarrow\!\!\!\!\rightarrow Y$, denoted $\mathrm{Gr}(f)$, is the set

$$\mathrm{Gr}(f) := \{(x,y) \in X \times Y \mid y \in f(x)\ \}.$$

Exercise: Draw $\mathrm{Gr}(f)$ for each of the three examples given above.

Continuity of Correspondences

Henceforth we assume that (X, d_X) and (Y, d_Y) are metric spaces. In order to develop the idea of continuity for correspondences, let's begin by recalling the definition of a continuous function:

Definition: A function $f : X \to Y$ is **continuous** at $\overline{x}$ if for every open set V such that $f(\overline{x}) \in V$, there is an open set U such that $\overline{x} \in U$ and $x \in U \Rightarrow f(x) \in V$.

For correspondences, "$f(\overline{x}) \in V$" and "$f(x) \in V$" have no meaning, because $f(\overline{x})$ and $f(x)$ are sets. There are two somewhat weaker notions of continuity for correspondences, called **upper hemicontinuity** and **lower hemicontinuity**, each of which is a partial analogue of continuity of a single-valued function. The two new notions are distinct in general, but if the correspondence is singleton-valued (in effect, a single-valued function), then each of the notions coincides with continuity of f.

Definition: A correspondence $f : X \longrightarrow\!\!\!\!\rightarrow Y$ is

(a) **upper hemicontinuous** (UHC) at $\overline{x} \in X$ if for every open set V such that $f(\overline{x}) \subseteq V$, there is an open set U such that $\overline{x} \in U$ and $x \in U \Rightarrow f(x) \subseteq V$;

(b) **lower hemicontinuous** (LHC) at $\overline{x} \in X$ if for every open set V such that $f(\overline{x}) \cap V \neq \emptyset$, there is an open set U such that $\overline{x} \in U$ and $x \in U \Rightarrow f(x) \cap V \neq \emptyset$.

(c) **continuous** at $\overline{x} \in X$ if it is both UHC and LHC at $\overline{x}$.

A correspondence is UHC/LHC/continuous on X if it is UHC/LHC/continuous at each $\overline{x} \in X$.

There is one more continuity property of correspondences that's very useful:

Definition: A correspondence $f : X \twoheadrightarrow Y$ has a *closed graph* if $\mathrm{Gr}(f)$ is a closed subset of $X \times Y$.

Remark: A correspondence thus has a closed graph if and only if it satisfies the following condition: if $\{x_n\}$ and $\{y_n\}$ are sequences in X and Y such that $y_n \in f(x_n)$ for each n, $\{x_n\} \to x$ and $\{y_n\} \to y$, then $y \in f(x)$.

Example 1 above has a closed graph and is continuous (both UHC and LHC), despite the fact that its target space is not compact. Example 2 is LHC but does not have a closed graph and is not UHC. Example 3 has a compact target space, has a closed graph, and is continuous (both UHC and LHC).

The closed graph property is more intuitive and usually much easier to work with than the UHC property. Moreover, in a great many applications the target space Y is compact, in which case any closed correspondence is UHC. We consider two more examples before proving this fact.

Example 4: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} [.3, .7] & \text{if } x \leqq \frac{1}{2} \\ \{\frac{1}{2}\} & \text{if } x > \frac{1}{2}. \end{cases}$$

This correspondence has a closed graph (and is therefore UHC, since the target space is compact), but it is not LHC.

Example 5: $f : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} \{\frac{1}{x}\} & \text{if } x > 0 \\ \{0\} & \text{if } x = 0. \end{cases}$$

This correspondence is singleton-valued (each of its images $f(x)$ is a singleton). If we regarded its images as points rather than sets (remove the curly brackets in the definition), f would be a single-valued function from $\mathbb{R}_+$ into $\mathbb{R}$, and clearly a discontinuous function — it is discontinuous at 0. As a correspondence it is nonempty-valued and has a closed graph, but is neither UHC nor LHC. (Note that its target space is not compact.)

Exercise: Draw Gr(f) for Examples 4 and 5, and verify the claims made above about the continuity properties Examples 1 to 5 have or don't have.

The following theorem is extremely useful, because in many applications a correspondence's target space is compact, or can be converted to a compact space.

Theorem: If Y is compact, then any correspondence $f : X \twoheadrightarrow Y$ that has a closed graph is UHC on X.

Proof: We assume that f has a closed graph but is not UHC at some $\overline{x} \in X$, and we will obtain a contradiction. Since f is not UHC at $\overline{x}$, there is an open set $V \subseteq Y$ such that $f(\overline{x}) \subseteq V$ but for every open ball of the form $B(\overline{x}, 1/n)$ there is an $x_n \in B(\overline{x}, 1/n)$ such that $f(x_n) \nsubseteq V$ — *i.e.*, such that some $y_n \in f(x_n)$ satisfies $y_n \notin V$.

Since Y is compact, $\{y_n\}$ has a convergent subsequence, which we also write as $\{y_n\}$, and we write $\overline{y} = \lim y_n$. Since f has a closed graph, we have $\overline{y} \in f(\overline{x})$. Since $\overline{y} \in f(\overline{x}) \subseteq V$ and V is open, there is an $\epsilon > 0$ such that $B(\overline{y}, \epsilon) \subseteq V$. Since $y_n \notin V$ for all n, we have $y_n \notin B(\overline{y}, \epsilon)$ for all n. But then $\{y_n\} \not\to \overline{y}$, a contradiction. $\|$

Example 5 demonstrates that compactness cannot be dispensed with in this theorem: if Y is not compact, a correspondence with a closed graph may not be UHC.

The following example demonstrates that the converse of the above theorem is false: even if Y is compact, a UHC correspondence need not have a closed graph.

Example 6: $f : [0,1] \twoheadrightarrow [0,1]$ is defined by $f(x) = (.3, .7)$ for all $x \in [0,1]$.

This correspondence is continuous (both UHC and LHC) and its target space is compact, but it does not have a closed graph.

Just as the closed graph property is useful because it can be characterized in terms of convergent sequences, there is a similar convergent-sequence characterization of lower hemicontinuity that is often easier to use than the definition of LHC.

Theorem: A correspondence $f : X \twoheadrightarrow Y$ is LHC at $x \in X$ if and only if for every sequence $\{x_n\} \to x$ and every $y \in f(x)$, there is a sequence $\{y_n\}$ that satisfies both (1) $y_n \in f(x_n)$ for all n and (2) $\{y_n\} \to y$.

Proof: de la Fuente, p. 111.

Fixed Points of Correspondences

Definition: A fixed point of a correspondence $f : X \twoheadrightarrow Y$ is an $x^* \in X$ for which $x^* \in f(x^*)$.

The most commonly used generalization of Brouwer's Theorem to correspondences is Kakutani's Fixed Point Theorem.

Kakutani's Theorem: Let $f : S \twoheadrightarrow S$ be a correspondence. If S is nonempty, compact, and convex, and if f is nonempty-valued, convex-valued, and has a closed graph, then f has a fixed point.

When we say that f is nonempty-valued and convex-valued, we mean that $f(x)$ is a nonempty convex set for every $x \in S$.

Example 6: $f : [0, 1] \twoheadrightarrow [0, 1]$ is defined by

$$f(x) = \begin{cases} [.6, .8] & \text{if } x < \frac{1}{2} \\ [.2, .8] & \text{if } x = \frac{1}{2} \\ [.2, .4] & \text{if } x > \frac{1}{2}. \end{cases}$$

This correspondence is convex-valued and has a closed graph, and it therefore has a fixed point. Its unique fixed point is $x^* = \frac{1}{2}$.

Example 7: $f : [0, 1] \twoheadrightarrow [0, 1]$ is defined by

$$f(x) = \begin{cases} \{.7\} & \text{if } x < \frac{1}{2} \\ [.2, .4] & \text{if } x \geqq \frac{1}{2}. \end{cases}$$

This correspondence has no fixed point. The correspondence is convex-valued but does not have a closed graph: it's discontinuous at $x = \frac{1}{2}$.

Example 8: $f : [0, 1] \twoheadrightarrow [0, 1]$ is defined by

$$f(x) = \begin{cases} [.6, .8] & \text{if } x < \frac{1}{2} \\ [.2, .4] \cup [.6, .8] & \text{if } x = \frac{1}{2} \\ [.2, .4] & \text{if } x > \frac{1}{2}. \end{cases}$$

This correspondence has no fixed point. The correspondence has a closed graph but is not convex-valued: it's convex-valued at every x except $x = \frac{1}{2}$.

The Maximum Theorem

Economic models of individual behavior generally assume optimizing behavior. We typically use the classical Weierstrass Theorem ("a continuous real-valued function on a compact set attains a maximum") to infer that there is actually an optimizing action available for the individual to choose. Is the individual's behavior continuous? That is, does the individual's chosen action respond continuously to changes in his environment? When we have a specific functional form for the objective function (*e.g.*, a Cobb-Douglas utility function), we can often obtain a closed-form expression for the behavioral function and determine directly whether it's continuous. Even when we can't obtain a closed-form behavioral function, if the objective function is differentiable we can generally apply the Implicit Function Theorem to establish continuity (and even differentiability) of the behavioral function.

The preceding paragraph implicitly assumed that the individual's behavior is described by a *single-valued* function — that the optimizing action is always unique. We now have correspondences at our disposal, so we can deal as well with situations in which the optimizing action is not unique — in which the behavioral function is actually a correspondence. The Maximum Theorem is used pervasively in economics and game theory to infer that a behavioral correspondence is UHC or has a closed graph.

In applications of the Maximum Theorem, the set X in the statement of the theorem below is typically the action space; E is the set of possible environments (the parameter space); the function u is the objective function; the correspondence φ describes how the set of available actions depends upon the environment; and μ is the behavioral correspondence, describing how the individual's actions depend upon the environment he faces.

In demand theory, for example, X would be the consumption set (or a compact subset of it); E would be the set of possible price-lists (and perhaps wealth/income levels); u would be the consumer's utility function; φ would be the correspondence that determines the budget set from the market prices; and μ would be the consumer's demand correspondence.

The Maximum Theorem: Let X be a compact subset of $\mathbb{R}^l$; let E be a subset of $\mathbb{R}^m$; let $u : X \times E \to \mathbb{R}$ be a continuous function; and let $\varphi : E \twoheadrightarrow X$ be a continuous correspondence. Then the correspondence $\mu : E \twoheadrightarrow X$ defined by

$$\mu(e) = \{\, x \in \varphi(e) \,|\, x \text{ maximizes } u(\cdot, e) \text{ on } \varphi(e) \,\}$$

is nonempty-valued and has a closed graph and the value function $v : E \to \mathbb{R}$ defined by $v(e) = \max\limits_{x \in \varphi(e)} u(x, e)$, or (in a slight abuse of notation) $v(e) = u(\mu(e), e)$, is continuous.

Proof: This is an obvious corollary of a slightly more general version of the theorem, given below.

Note that the domain of the objective function u includes not only actions (in the demand theory application these are consumption bundles), but parameters as well (in the demand application these are price-lists). This seems odd at first glance. There are three observations to make about this:

(1) In the demand theory application we typically assume that u is constant with respect to the parameters $e \in E$ — the prices. So for this application a Maximum Theorem in which u depends only upon consumption bundles in X would be just fine.

(2) Including the prices as arguments of u does, however, allow us to analyze consumers whose utility depends upon prices as well as consumption levels.

(3) The theory of the firm is an example of an application where we need to have the objective function u depend upon the parameters. In this application, the elements of X are the firm's feasible production plans, *i.e.*, input-output combinations. The elements of E are again price-lists. The objective function u is the firm's profit function — and note that profit does depend upon both the firm's choice of production plan (in X) and the market prices (in E). The correspondence φ describes how the available plans depend upon prices — typically, this is assumed to be a constant correspondence in the theory of the firm: the production plans available to the firm depend upon its technological capabilities but not upon prices. And of course the correspondence μ describes how the firm's profit-maximizing choices of input and output levels depend upon the market prices — the firm's supply correspondence for outputs and its demand correspondence for inputs.

Here's a slightly more general version of the Maximum Theorem, in which the action space X needn't be compact (although note that the feasible-set correspondence φ must be compact-valued):

The Maximum Theorem: Let X be a subset of $\mathbb{R}^l$; let E be a subset of $\mathbb{R}^m$; let $u : X \times E \to \mathbb{R}$ be a continuous function; and let $\varphi : E \twoheadrightarrow X$ be a continuous and compact-valued correspondence. Then the correspondence $\mu : E \twoheadrightarrow X$ defined by

$$\mu(e) = \{\ x \in \varphi(e) \,|\, x \text{ maximizes } u(\cdot, e) \text{ on } \varphi(e)\ \}$$

is nonempty-valued and UHC and the value function $v : E \to \mathbb{R}$ defined by $v(e) = \max\limits_{x \in \varphi(e)} u(x, e)$ is continuous.

Proof: de la Fuente, p. 301.

Some useful facts about correspondences are provided here. The proofs are straightforward and are good exercises for understanding correspondences.

Theorem: If Y is compact and the correspondences $f : X \twoheadrightarrow Y$ and $g : X \twoheadrightarrow Y$ both have closed graphs, then the sum $f + g$ also has a closed graph, where $f + g$ is the correspondence defined by

$$(f + g)(x) := \{\ y_1 + y_2 \in Y \mid y_1 \in f(x) \quad \text{and} \quad y_2 \in g(x)\ \}.$$

Theorem: If Y and Z are compact and the correspondences $f : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$ both have closed graphs, then the composition $f \circ g$ also has a closed graph, where $f \circ g$ is the correspondence defined by

$$\begin{aligned}(f \circ g)(x) := g(f(x)) &= \{\ z \in Z \mid \exists y \in Y : y \in f(x) \ \&\ z \in g(y)\ \} \\ &= \bigcup \{\ g(y) \mid y \in f(x)\ \}.\end{aligned}$$