

Continuous Functions in Metric Spaces

Throughout this section let (X, d_X) and (Y, d_Y) be metric spaces.

Definition: Let $\bar{x} \in X$. A function $f : X \rightarrow Y$ is continuous at \bar{x} if for every sequence $\{x_n\}$ that converges to \bar{x} , the sequence $\{f(x_n)\}$ converges to $f(\bar{x})$.

Definition: A function $f : X \rightarrow Y$ is continuous if it is continuous at every point in X .

Theorem: A function $f : X \rightarrow Y$ is continuous at \bar{x} if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d_X(x, \bar{x}) < \delta \Rightarrow d_Y(f(x), f(\bar{x})) < \epsilon$ — *i.e.*,

$$\forall \epsilon > 0 : \exists \delta > 0 : x \in B(\bar{x}, \delta) \Rightarrow f(x) \in B(f(\bar{x}), \epsilon). \quad (*)$$

Proof:

(\Rightarrow): Let $\epsilon > 0$. Suppose, by way of contradiction, that there is no $\delta > 0$ such that $d_X(x, \bar{x}) < \delta \Rightarrow d_Y(f(x), f(\bar{x})) < \epsilon$ — *i.e.*,

$$\forall \delta > 0 : \exists x \in B(\bar{x}, \delta) \text{ for which } f(x) \notin B(f(\bar{x}), \epsilon).$$

Then, in particular, for every $n \in \mathbb{N}$, let $\frac{1}{n}$ play the role of δ above: there is an $x_n \in B(\bar{x}, \frac{1}{n})$ for which $f(x_n) \notin B(f(\bar{x}), \epsilon)$. We therefore have a sequence $\{x_n\}$ in X that converges to \bar{x} but the sequence $\{f(x_n)\}$ does not converge to $f(\bar{x})$, contradicting our assumption that f is continuous.

(\Leftarrow): Assume that $(*)$ holds, and let $\{x_n\}$ be a sequence that converges to \bar{x} . In order to show that $\{f(x_n)\}$ converges to $f(\bar{x})$, let $\epsilon > 0$. According to $(*)$, there is a $\delta > 0$ for which

$$x \in B(\bar{x}, \delta) \Rightarrow f(x) \in B(f(\bar{x}), \epsilon).$$

Since $\{x_n\} \rightarrow \bar{x}$, we can choose $\bar{n} \in \mathbb{N}$ such that $n > \bar{n} \Rightarrow x_n \in B(\bar{x}, \delta)$. But then

$$n > \bar{n} \Rightarrow f(x_n) \in B(f(\bar{x}), \epsilon);$$

i.e., $\{f(x_n)\}$ converges to $f(\bar{x})$, and f is therefore continuous at \bar{x} . □

Remark: For functions f from \mathbb{R}^n to \mathbb{R}^m this theorem says that f is continuous at $\bar{x} \in \mathbb{R}^n$ if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|x - \bar{x}\| < \delta \Rightarrow \|f(x) - f(\bar{x})\| < \epsilon$.

Theorem: A function $f : X \rightarrow Y$ is continuous if and only if for every open set V in Y the inverse image $f^{-1}(V)$ is an open set in X .

Proof: Exercise.

An elementary consequence of the preceding theorem is its analogue in terms of closed sets:

Theorem: A function $f : X \rightarrow Y$ is continuous if and only if for every closed set S in Y the inverse image $f^{-1}(S)$ is a closed set in X .

This gives us four equivalent definitions of a continuous function f from X to Y :

If for every sequence $\{x_n\}$ that converges to \bar{x} , the sequence $\{f(x_n)\}$ converges to $f(\bar{x})$.

If for every $\bar{x} \in X : \forall \epsilon > 0 : \exists \delta > 0 : x \in B(\bar{x}, \delta) \Rightarrow f(x) \in B(f(\bar{x}), \epsilon)$.

If the inverse image of any open set in Y is an open set in X .

If the inverse image of any closed set in Y is a closed set in X .

Remark: We've already seen applications of these ideas to preferences and utility functions, and to the possibility of representing a preference by a utility function.

Remark: When the target space Y is actually a normed vector space, we can develop the algebra of limits and of continuous functions. For example, we can show that the sum of two continuous functions is continuous, that the product of two continuous functions is continuous, etc.

In Euclidean space — \mathbb{R}^n with any norm — we say that a set is **compact** if it is both closed and bounded. One of the most important properties of continuous functions is that they “preserve” compactness — *i.e.*, if X is a compact subset of \mathbb{R}^n and if $f : X \rightarrow \mathbb{R}^m$ is a continuous function, then the image of X , $f(X)$, is a compact set in \mathbb{R}^m . This is the Weierstrass Theorem. In fact, the Weierstrass Theorem holds in general metric spaces:

Weierstrass Theorem: If X is compact and $f : X \rightarrow Y$ is continuous, then $f(X)$ is a compact subset of Y .

Corollary: If $f : X \rightarrow \mathbb{R}$ is a continuous real-valued function on a compact set, then f attains a maximum and a minimum on X .

Instead of proving the Weierstrass Theorem here, we defer the proof until after we've developed our next important concept, the Bolzano-Weierstrass (B-W) Property. There are two good reasons for waiting until then to do the proof: (1) we need the B-W Property in order to generalize the notion of a compact set to general metric spaces, and (2) the theorem's proof is *much* easier using the B-W Property in the general setting than if we were to do it using the closed-and-bounded definition of compactness in Euclidean space.