Cauchy Sequences and Complete Metric Spaces

Definition: A sequence $\{x_n\}$ in a metric space (X, d) is **Cauchy** if

$$\forall \epsilon > 0 : \exists \bar{n} \in \mathbb{N} : \; m, n > \bar{n} \Rightarrow d(x_m, x_n) < \epsilon.$$

Remark: Every convergent sequence is Cauchy.

Proof:

Let $\{x_n\} \to \bar{x}$, let $\epsilon > 0$, let $\bar{n}$ be such that $n > \bar{n} \Rightarrow d(x_n, \bar{x}) < \epsilon/2$, and let $m, n > \bar{n}$. Then

$$d(x_m, \bar{x}) < \frac{\epsilon}{2} \text{ and } d(x_n, \bar{x}) < \frac{\epsilon}{2},$$

and the Triangle Inequality yields

$$d(x_m, x_n) \leqq d(x_m, \bar{x}) + d(x_n, \bar{x}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square$$

Exercise: The real sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$ converges, so it is Cauchy. Prove directly that it is Cauchy, by showing how the $\bar{n}$ depends upon ϵ.

Example 1: Let $x_n = \frac{1}{n}\sqrt{2}$ for each $n \in \mathbb{N}$. Note that each x_n is an irrational number (*i.e.*, $x_n \in \mathbb{Q}^c$) and that $\{x_n\}$ converges to 0. Thus, $\{x_n\}$ converges in $\mathbb{R}$ (*i.e.*, to an element of $\mathbb{R}$). But 0 is a rational number (thus, $0 \notin \mathbb{Q}^c$), so although the sequence $\{x_n\}$ is entirely in $\mathbb{Q}^c$, it does not converge in $\mathbb{Q}^c$. Note, however, that $\{x_n\}$ is Cauchy.

Example 2: Let $\bar{x}$ be an irrational number, and for each $n \in \mathbb{N}$ let $\{x_n\}$ be a rational number in the interval $\left(\bar{x} - \frac{1}{n}, \bar{x} + \frac{1}{n}\right)$. Then $\{x_n\}$ is a sequence of rational numbers that converges to the irrational number $\bar{x}$ — *i.e.*, each $\{x_n\}$ is in $\mathbb{Q}$ and $\{x_n\} \to \bar{x} \notin \mathbb{Q}$. Thus, in a parallel to Example 1, $\{x_n\}$ here is a Cauchy sequence in $\mathbb{Q}$ that does not converge in $\mathbb{Q}$.

Examples 1 and 2 demonstrate that both the irrational numbers, $\mathbb{Q}^c$, and the rational numbers, $\mathbb{Q}$, are not entirely well-behaved metric spaces — they are not *complete* in that there are Cauchy sequences in each space that don't converge to an element of the space.

Definition: A metric space (X, d) is **complete** if every Cauchy sequence in X converges in X (*i.e.*, to a limit that's in X).

Example 3: The real interval $(0, 1)$ with the usual metric is not a complete space: the sequence $x_n = \frac{1}{n}$ is Cauchy but does not converge to an element of $(0, 1)$.

Example 4: The space $\mathbb{R}^n$ with the usual (Euclidean) metric is complete. (We have not yet shown this.)

Example 5: The space $C[0,1]$ is complete. (We have not yet shown this.)

Example 6: The closed unit interval $[0,1]$ with the usual metric is a complete metric space. This is easy to prove, using the fact that $\mathbb{R}$ is complete.

Exercise: In a previous exercise set we worked with a sequence of distribution functions F_n defined by

$$F_n(x) = \begin{cases} nx, & \text{if } x \leqq \frac{1}{n} \\ 1\,, & \text{if } x \geqq \frac{1}{n}. \end{cases}$$

on the unit interval $[0,1]$ in $\mathbb{R}$. We showed that $\{F_n\}$ does not converge in $C[0,1]$. Therefore, if $\{F_n\}$ were Cauchy, $C[0,1]$ would not be complete. Verify that $\{F_n\}$ is *not* Cauchy.

Example 7: (Obtaining $\mathbb{R}$ as the completion of $\mathbb{Q}$.)
Let S be the set of Cauchy sequences in $\mathbb{Q}$ — *i.e.*, the set of Cauchy sequences of rational numbers — with the usual metric. Define a relation $\sim$ on S as follows:

$$\{x_n\} \sim \{x_n'\} \text{ if } \forall \epsilon > 0 : \exists \bar{n} \in \mathbb{N} : m, n > \bar{n} \Rightarrow d(x_m, x_n') < \epsilon$$

Let $\mathbb{Q}^* = S/\sim$, the partition of S consisting of equivalence classes of Cauchy sequences. Define the distance function d^* for $\mathbb{Q}^*$ as follows:

For any $x, x' \in \mathbb{Q}^*$, let $\{x_n\} \in x$ and $\{x_n'\} \in x'$ (*i.e.*, $x = [\{x_n\}]$ and $x' = [\{x_n'\}]$.

Then define $d^*(x,x')$ by $d^*(x,x') = \lim_{n\to\infty} d(x_n, x_n')$.

It's pretty straightforward to show that d^* is well-defined and is a metric for $\mathbb{Q}^*$. The metric space $(\mathbb{Q}^*, d^*)$ can be placed into one-to-one correspondence with $(\mathbb{R}, |\cdot|)$, each constant sequence $\{r, r, r, \ldots\}$ of rationals corresponding to the rational number $r \in \mathbb{Q} \subseteq \mathbb{R}$. The set $\mathbb{Q}^*$ is one definition of $\mathbb{R}$.

Exercise: Verify that the relation $\sim$ defined in Example 7 is an equivalence relation.

Example 7: (The Completion of a Metric Space)
Let (X,d) be a metric space that is not complete. Just as in Example 7, let S be the set of Cauchy sequences in X; define the equivalence relation $\sim$ in the same way, and let X^* be the quotient space $S/\sim$; and define d^* on the quotient space X^* in the same way as in Example 7. Then we can show, just as in Example 7, that d^* is well-defined and is a metric for X^*; that (X^*, d^*) is a complete metric space; and that X corresponds to a subset of X^* — we say that X **is embedded in** X^*. The complete metric space (X^*, d^*) is called the **completion** of (X,d).

Example 8: The open unit interval $(0,1)$ in $\mathbb{R}$, with the usual metric, is an incomplete metric space. What is its completion, $((0,1)^*, d^*))$?

Theorem: A subset of a complete metric space is itself a complete metric if and only if it is closed.

Proof: Exercise.

Recall that every normed vector space is a metric space, with the metric $d(x, x') = \|x - x'\|$. Therefore our definition of a complete metric space applies to normed vector spaces: an n.v.s. is complete if it's complete as a metric space, *i.e.*, if all Cauchy sequences converge to elements of the n.v.s.

Definition: A complete normed vector space is called a **Banach space**.

Example 9: $\mathbb{R}^n$ with the Euclidean norm is a Banach space.

Example 10: $C[0, 1]$ is a Banach space.

Example 11: The unit interval $[0, 1]$ is a complete metric space, but it is not a Banach space because it is not a vector space.