The Bolzano-Weierstrass Property and Compactness

Definitions: Let S be a subset of $\mathbb{R}$. An **upper bound** of S is a number b such that $x \leqq b$ for every $x \in S$. A **least upper bound** of S is a b^* such that $b^* \leqq b$ for every b that's an upper bound of S.

Remark: In $\mathbb{R}$, a set can have no more than one least upper bound, so it makes sense to talk about *the* least upper bound of S. It is also called the *supremum* of S, denoted $\sup S$ or $\operatorname{lub} S$.

Lower bound, greatest lower bound, glb and inf are defined analogously.

Definition: A partially ordered set X has the **LUB Property** if every nonempty set that has an upper bound has a least upper bound.

The Completeness Axiom: $\mathbb{R}$ has the LUB property — any nonempty set of real numbers that has an upper bound has a least upper bound.

We've elsewhere used "complete" to mean that every Cauchy sequence converges (to a point in the set).

Remark: Assuming the Completeness Axiom in $\mathbb{R}$, one can prove that every Cauchy sequence converges (see below). Conversely, assuming that every Cauchy sequence converges in $\mathbb{R}$, one can prove the Completeness Axiom, *i.e.*, that $\mathbb{R}$ has the LUB Property.

The Monotone Convergence Theorem: Every bounded monotone sequence in $\mathbb{R}$ converges to an element of $\mathbb{R}$.

Proof: Let $\{x_n\}$ be a monotone increasing sequence of real numbers. Since it's bounded, it has a least upper bound b. We will show that $\{x_n\} \to b$. Suppose $\{x_n\}$ doesn't converge to b. Then for every $\epsilon > 0$ infinitely many terms of the sequence satisfy $|x_n - b| \geqq \epsilon$ — *i.e.*, $x_n \leqq b - \epsilon$. (x_n cannot be greater than b if b is an upper bound.) It follows that $x_n \leqq b - \epsilon$ for *all* $n \in \mathbb{N}$: since $\{x_n\}$ is increasing, if $x_m > b - \epsilon$ for some m, then $x_m > b - \epsilon$ for all larger n, contradicting that $x_n \leqq b - \epsilon$ for infinitely many n. Thus we have $x_n \leqq b - \epsilon$ for all $n \in \mathbb{N}$; *i.e.*, $b - \epsilon$ is an upper bound of $\{x_1, x_2, \ldots\}$, and therefore b is not a least upper bound of $\{x_1, x_2, \ldots\}$, a contradiction. Therefore $\{x_n\}$ does converge to b. $\square$

Theorem: $\mathbb{R}$ is a complete metric space (in fact, a complete normed vector space, or Banach space) — *i.e.*, every Cauchy sequence of real numbers converges.

Proof: See Simon & Blume, pages 805-806. Their proof uses the Completeness Axiom. We can also go the other direction, using the convergence of Cauchy sequences to prove the Completeness Axiom.

The Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.

Equivalent: Every sequence in a closed and bounded set has a convergent subsequence (i.e., the subsequence converges to an element of the set). This is equivalent because (a) every bounded sequence is contained in a closed and bounded set, and (b) every sequence in a closed and bounded set is bounded, and the limit of a convergent sequence in a closed set is itself in the set.

Proof of the theorem: Let $\{x_n\}$ be a bounded sequence and without loss of generality assume that every term of the sequence lies in the interval $[0,1]$. Divide $[0,1]$ into two intervals, $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. (Note: this is not a partition of $[0,1]$.) At least one of the halves contains infinitely many terms of $\{x_n\}$; denote that interval by I_1, which has length $\frac{1}{2}$, and let $\{x_{n'}\}$ be a subsequence of $\{x_n\}$ every term of which lies in I_1 (we can do this because I_1 contains infinitely many terms of $\{x_n\}$).

Now divide I_1 into two halves, each of length $(\frac{1}{2})^2$, at least one of which contains infinitely many terms of the (sub)sequence $\{x_{n'}\}$, and denote that half by I_2. Let $\{x_{n''}\}$ be a subsequence of $\{x_{n'}\}$ all of whose terms lie in I_2. Continuing in this way, we construct a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \ldots$, each of length $(\frac{1}{2})^n$, and each containing an infinite number of terms of the original sequence $\{x_n\}$. Finally, we construct a subsequence $\{z_n\}$ of $\{x_n\}$ made up of one term from each interval I_n. This subsequence is clearly Cauchy: for any N, $m,n > N \Rightarrow |z_m - z_n| < (\frac{1}{2})^N$. Therefore the subsequence $\{z_n\}$ converges, according to the Cauchy-sequence version of the Completeness Axiom. $\square$

The Bolzano-Weierstrass Theorem is true in $\mathbb{R}^n$ as well:

The Bolzano-Weierstrass Theorem: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof: Let $\{x^m\}$ be a bounded sequence in $\mathbb{R}^n$. (We use superscripts to denote the terms of the sequence, because we're going to use subscripts to denote the components of points in

$\mathbb{R}^n$.) The sequence $\{x_1^m\}$ of first components of the terms of $\{x^m\}$ is a bounded real sequence, which has a convergent subsequence $\{x_1^{m_k}\}$, according to the B-W Theorem in $\mathbb{R}$. Let $\{x^{m_k}\}$ be the corresponding subsequence of $\{x^m\}$. Then the sequence $\{x_2^{m_k}\}$ of second components of $\{x^{m_k}\}$ is a bounded sequence of real numbers, so it too has a convergent subsequence, and we again have a corresponding subsequence of $\{x^{m_k}\}$ (and therefore of $\{x^m\}$), in which the sequences of first and second components both converge. Continuing for n iterations, we end up with a subsequence $\{z^m\}$ of $\{x^m\}$ in which the sequences of first, second, $\ldots$, nth components all converge, and therefore the subsequence $\{z^m\}$ itself converges in $\mathbb{R}^n$. $\square$

As described above, it's elementary to show that the following form of the B-W Theorem is equivalent to the one we've just proved:

The Bolzano-Weierstrass Theorem: Every sequence in a closed and bounded subset of $\mathbb{R}^n$ has a convergent subsequence. (And since the subset is closed, the subsequence must converge to a point that's in the subset).

This is so important that we elevate it to a definition:

Definition: A set S in a metric space has the **Bolzano-Weierstrass Property** if every sequence in S has a convergent subsequence — *i.e.*, has a subsequence that converges to a point in S.

The B-W Theorem states that closed and bounded (*i.e.*, compact) sets in $\mathbb{R}^n$ have the B-W Property. We can also prove the converse of the B-W Theorem, that any set in $\mathbb{R}^n$ with the B-W Property is closed and bounded (compact). See Theorem 29.6 of Simon & Blume.

In other words, the compact sets in $\mathbb{R}^n$ are *characterized* by the B-W Property.

So how do things work in general metric spaces? Are compact sets characterized by B-W?
What is the definition of a compact set in a metric space?
Let's say, tentatively, that it's still defined as a closed and bounded set.
Are closed and bounded sets still characterized by B-W in metric spaces?

Example: Let (X, d) be an infinite set with the discrete metric: $x \neq x' \Rightarrow d(x, x') = 1$. (For example, let $X = \mathbb{N} = \{1, 2, \ldots\}$.) Of course the space is closed. And the space is bounded: let x^* be any element of X; every element $x \in X$ except x^* is at distance $d(x, x^*) = 1$ from

x^*. Let $\{x_n\}$ be any sequence of distinct points in X; the sequence is bounded. Moreover, it does not have a convergent subsequence (it consists of distinct points, all of them equidistant from one another). So closed and bounded sets in a metric space don't necessarily have the B-W Property — in this respect, closed and bounded sets in some metric spaces will behave very differently than compact sets in $\mathbb{R}^n$. **Exercise:** Is the sequence described in the example Cauchy? Describe all Cauchy sequences and all convergent sequences in this metric space.

The essential feature of compact sets in $\mathbb{R}^n$ is that they have the B-W Property. Lots of other properties of compact sets follow from that — for example, the Weierstrass Theorem, that a continuous real-valued function on a compact set attains a maximum and a minimum. But here we see that closed and bounded sets in an arbitrary metric space may not have the B-W property; therefore we don't want to call them compact. Instead, we simply *define* compact sets to be the ones that have the B-W Property.

Definition: A metric space is **compact** if it has the B-W Property.

Let's review: In $\mathbb{R}^n$ we called the closed and bounded sets compact, and they were characterized by the B-W Property. In metric spaces we have definitions of closed sets and bounded sets, but closed and bounded sets don't necessarily have the B-W Property. So we *defined* compact sets to be the ones that have the B-W Property — and compactness is therefore still characterized by the B-W Property, but here it's true by definition.

Note: You will also see a definition that says a compact set is one that has the Heine-Borel Property – every open cover has a finite subcover. (We may study this later.) Just as "closed and bounded" didn't get us what we wanted when we went from $\mathbb{R}^n$ to a metric space, the B-W Property doesn't get us what we want when we go to a topological space (a space where open and closed sets are the defining concepts but the space may not have a metric structure).

Theorem: Let S be a compact set in a metric space. Then
(a) S is closed; (b) S is bounded; (c) S is complete.

Theorem: A closed subset of a compact metric space is compact.

Theorem: Let $f : X \to Y$. If X is compact and f is continuous, then $f(X)$ is compact.

Proof: We must show that $f(X)$ has the B-W Property. Let $\{y_n\}$ be a sequence in $f(X)$; we must show that $\{y_n\}$ has a convergent subsequence. For each $n \in \mathbb{N}$, let $x_n \in X$ be such that $f(x_n) = y_n$ (which we can do because $y_n \in f(X)$). Since X is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to some $\overline{x} \in X$. Since f is continuous, $\{f(x_{n_k})\}$ converges to $f(\overline{x}) \in Y$. Since $\overline{x} \in X$, we have $f(\overline{x}) \in f(X)$. $\square$