Binary Relations

Definition: A **binary relation** between two sets X and Y (or between the elements of X and Y) is a subset of $X \times Y$ — *i.e.*, is a set of ordered pairs $(x, y) \in X \times Y$.

If R is a relation between X and Y (*i.e.*, if $R \subseteq X \times Y$), we often write xRy instead of $(x, y) \in R$. We write R^c for the complement of R — *i.e.*, $xR^c y$ if and only if $(x, y) \notin R$. If X and Y are the same set, so that the relation R is a subset of $X \times X$, we say that R is a relation on X.

Example 1:

X is a set of students, say $X = \{Ann, Bev, Carl, Doug\}$.

Y is a set of courses, say $Y = \{History, Math, Economics\}$.

Then $X \times Y$ has 12 elements. An example of a relation $R \subseteq X \times Y$ is the set of pairs (x, y) for which "x is enrolled in y." Another example is the relation $\widetilde{R}$ defined by "$x\widetilde{R}y$ if x received an A grade in y". In this example we would likely have $\widetilde{R} \subseteq R$, *i.e.*, $x\widetilde{R}y \Rightarrow xRy$.

The following example defines two important relations associated with any function $f : X \to Y$.

Example 2:

(a) Define G as follows: $G := \{\ (x, y) \in X \times Y \mid y = f(x)\ \}$. Clearly, G is simply the graph of the function f. But we can just as well regard G as a relation between members of X and Y, where xGy means that $y = f(x)$. Note that $G \subseteq X \times Y$.

(b) Define $x \sim x'$ to mean $f(x) = f(x')$. Note that $x \sim x$ for every $x \in X$; that if $x \sim x'$ then $x' \sim x$; and that if $x \sim x'$ and $x' \sim x''$, then $x \sim x''$. Any relation with these three properties is called an *equivalence relation*. Equivalence relations are important; we'll see a lot more of them shortly. Note that this relation is a subset of $X \times X$.

Example 3:

Let X be an arbitrary set and let $u : X \to \mathbb{R}$ be a real-valued function on X. If X is interpreted as a set of *alternatives* and u is interpreted as a *utility function* that represents someone's preference over the alternatives, then we interpret $u(x') > u(x)$ to mean the person strictly prefers x' to x and we define the corresponding relation P on X by $x'Px \Leftrightarrow u(x') > u(x)$. Similarly, we interpret $u(x') = u(x)$ to mean the person is indifferent between x' and x, and we define the relation I by $xIy \Leftrightarrow u(x') = u(x)$; and we interpret $u(x') \geqq u(x)$ to mean the person weakly prefers x' to x (she either prefers x' or is indifferent), and we define the relation R by $xRy \Leftrightarrow u(x') \geqq u(x)$. It's common to use $x' \succ x$ instead of $x'Px$; $x' \sim x$ instead of $x'Ix$; and $x' \succsim x$ instead of $x'Rx$.

The set X in Example 3 could be a set of consumption bundles in $\mathbb{R}^n$, as in demand theory, but that's not necessary; X could be any set of alternatives over which someone has preferences.

Note that the indifference relation I, or $\sim$, in Example 3 is the same relation defined in Example 2(b). It therefore has the three properties described there and is an equivalence relation. The strict preference relation P, or $\succ$, has the third property but not the other two; and the weak preference relation R, or $\succsim$, has the first and third property but not the second. These properties, and several others, are important enough that we give them names and define them formally:

Definitions: A binary relation R on a set X is

(a) **reflexive** if $\forall x \in X: \; xRx$;

(b) **symmetric** if $\forall x, x' \in X: \; x'Rx \Rightarrow xRx'$;

(c) **transitive** if $\forall x, x', x'' \in X: \; [x''Rx' \; \& \; x'Rx] \Rightarrow x''Rx$;

(d) **complete** if $\forall x, x' \in X: \; xRx'$ or $x'Rx$;

(e) **antisymmetric** if $\forall x, x' \in X: \; [x'Rx \; \& \; xRx'] \Rightarrow x = x'$

(f) **asymmetric** if $\forall x, x' \in X: \; [x'Rx \;\; \Rightarrow \;\; xR^c x']$

(g) **irreflexive** if $\forall x \in X: \; xR^c x$.

Example 4:

X is a set of people. Each of the following is a binary relation on X:

(a) $xNy:$ x lives next door to y. N would typically be symmetric, irreflexive, and not transitive.

(b) $xBy:$ x lives on the same block as y. B would typically be reflexive, symmetric, and transitive.

(c) $xSy:$ x is a sister of y. S would typically be irreflexive, not symmetric (unless all elements of X are female), and not transitive.

(d) $xAy:$ x is an ancestor of y. A would typically be irreflexive, asymmetric, and transitive — a strict preorder, as we'll define shortly.

(e) $xDy:$ x is a daughter of y. D would be irreflexive, asymmetric, and not transitive.

Example 3 continued:

Note that the relation $\succsim$ is complete and the relations $\succ$ and $\sim$ are typically not complete. The relations $\succsim$ and $\sim$ are typically not antisymmetric; $\succ$ is vacuously antisymmetric. ("Vacuous" because the antecedent in the definition, $x' \succ x \; \& \; x \succ x'$, can never be satisfied.) Can you construct special cases of Example 3 in which $\succ$ or $\sim$ are complete? How about special cases in which $\succsim$ or $\sim$ are antisymmetric? (Example 5 may be helpful here.)

Example 5:

The usual ordering of the real numbers in $\mathbb{R}$, in which $\geqq$ is the weak ordering and $>$ is the strict ordering, is analogous to the relations defined in Example 3, but generally not quite the same. For example, $\geqq$ *is* antisymmetric, and so is the equality relation, $=$, unlike $\succsim$ and $\sim$.

Examples 3 and 5 display the difference between an **ordering** of a set and what we call a **pre-ordering** of a set: if $\succsim$ is merely a preorder but not an order, then two or more *distinct* elements x and x' can satisfy both $x' \succsim x$ and $x \succsim x'$ (for example, two consumption bundles x and x' between which someone is indifferent). But if $\succsim$ is an order (often called a *total order*), that can't happen — $x' \succsim x$ and $x \succsim x'$ require that x and x' be the same element.

Definition: A relation R on a set X is

(a) a **preorder** if it is transitive and either reflexive (a **weak preorder**) or irreflexive (a **strict preorder**);

(b) an **order** if it is complete, transitive, and antisymmetric.

Definition: If $\succsim$ is a preorder on X, then

(a) the associated strict preorder, denoted $\succ$, is defined by $x' \succ x \;\Leftrightarrow\; [x' \succsim x \;\&\; x \not\succsim x']$;

(b) the associated equivalence relation $\sim$ is defined by $x' \sim x \;\Leftrightarrow\; [x' \succsim x \;\&\; x \succsim x']$.

Remark: The terminology in the above definition is appropriate: $\succ$ is indeed a strict preorder and $\sim$ is an equivalence relation.

Exercise: Provide a proof of this remark.

In economics and decision theory we're often interested in elements that are *best*, or a *maximum*, in X according to a preorder $\succsim$ — *i.e.*, an element that's at least as good (or at least as large) as every other element in X. If a preorder is not complete, there may not be a maximum element, so we also define the weaker notion of a *maximal* element. (Do *not* use Definition 3.7 in de la Fuente.)

Definition: If $\succsim$ is a preorder on X, then

(a) $\widehat{x}$ is a **maximum** element for $\succsim$ if $\forall x \in X: \; \widehat{x} \succsim x$;

(b) $\widehat{x}$ is a **maximal** element for $\succsim$ if $\nexists x \in X: \; x \succ \widehat{x}$.

Similarly, $\widehat{x}$ is a **minimum** or **minimal** element if $\forall x \in X: \; x \succsim \widehat{x}$ or $\nexists x \in X: \; \widehat{x} \succ x$.

Exercise: Prove that for any preorder $\succsim$, a maximum element is also a maximal element, and prove that if $\succsim$ is complete then any maximal element is also a maximum element.

Definition: If R is a binary relation on X and if $\bar{x} \in X$,

(a) the R-**upper-contour set** of $\bar{x}$ is the set $R\bar{x} = \{x \in X \mid xR\bar{x}\}$, and

(b) the R-**lower-contour set** of $\bar{x}$ is the set $\bar{x}R = \{x \in X \mid \bar{x}Rx\}$.

According to this definition, if $\succsim$ is a preorder on X we would use $\succsim \bar{x}$ and $\succ \bar{x}$ for the weak and strict upper-contour sets of $\bar{x}$. But I find it's generally better to use the notation $R\bar{x}$ and $P\bar{x}$ for the weak and strict $\succsim$-upper-contour sets, and to use $\bar{x}R$ and $\bar{x}P$ for the weak and strict lower-contour sets.

Exercise: Define the relation $\succsim$ on $\mathbb{R}^2$ by $\mathbf{x}' \succsim \mathbf{x} \;\Leftrightarrow\; [x_1' \geqq x_1 \;\&\; x_2' \geqq x_2]$.

(a) In a diagram, depict the weak upper- and lower-contour sets of a typical point $\bar{\mathbf{x}} \in \mathbb{R}^2$.

(b) Prove that $\succsim$ is a preorder.

(c) Is $\succsim$ complete? Is it antisymmetric? What is the equivalence class $[\bar{\mathbf{x}}]$ of a typical $\bar{\mathbf{x}} \in \mathbb{R}^2$?

(d) Let $c \in \mathbb{R}$ be an arbitrary real number, and let $X = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 \leqq c\}$. Is a maximal element for $\succsim$ on X also a maximum element? If yes, provide a proof; if not, provide an example of a maximal element that is not a maximum.

A preorder is a natural, intuitively appealing way to represent someone's decision behavior: when faced with a set X of available alternatives, we assume that his choice will be an element $\widehat{x}$ to which no other element x is strictly preferred — *i.e.*, it will be a maximal element of X with respect to the decision-maker's preference $\succsim$. But relations are cumbersome and awkward to work with; functions are much more tractable analytically. So if we have a given preorder $\succsim$ on a set X, we would like to be able to transform it into a utility function u on X in such a way that u and $\succsim$ are related as in Example 3.

Definition: Let R be a binary relation on a set X. A real-valued function $u : X \to \mathbb{R}$ is a **utility function** for R, or a **representation** of R, if

$$\forall x, x' \in X : \; u(x') \geqq u(x) \Leftrightarrow x'Rx.$$

R is said to be **representable** if there is a utility function for R.

Compare this definition to Example 3. Note first of all that if the relation R is representable, then according to Example 3 it must be a preorder — in fact, a *complete* preorder.

Second, note that in Example 3 we went in the opposite direction to the one indicated in the above definition: we started with the function u as the representation of someone's preference and defined the associated preorder $\succsim$. But it's actually preferences that we think are fundamental, not utility functions, so we would like to know when we can find a utility function to represent a given preorder. We'll return to this idea later and provide conditions on a preorder $\succsim$ that are sufficient to ensure that it can be represented by a utility function. You'll see this idea in Economics 501A as well.

Equivalence Relations and Partitions

In Examples 2 and 3 we encountered the idea of an equivalence relation, although we didn't single it out with its own formal definition. Equivalence relations are one of the most ubiquitous and fundamental ideas in mathematics, and we'll encounter them over and over again in this course. The notation $\sim$ that we used in Examples 2 and 3 is the standard notation for an equivalence relation.

Definition: An **equivalence relation** on a set X is a binary relation that is reflexive, symmetric, and transitive. Equivalence relations are typically denoted by the symbol $\sim$. The set $\{x \in X \mid x \sim \bar{x}\}$ of all members of X that are equivalent to a given member $\bar{x}$ is called the **equivalence class** of $\bar{x}$ and is typically written $[\bar{x}]$. If $x \sim x'$, we say that "x is equivalent to x'."

Example 5 continued:
X is a set of people. The relation "is the same age as" is an equivalence relation. The relation "is a brother of" is not. The relation "lives in the same house as" is an equivalence relation. The relation "lives next door to" is not.

Example 3 continued:
The equivalence class $[\bar{x}]$ of an alternative $\bar{x} \in X$ is the set of all the alternatives that are "indifferent to $\bar{x}$" — *i.e.*, all the alternatives x for which the decision-maker is indifferent between x and $\bar{x}$. If the set X is $\mathbb{R}^2_+$ and the utility function u is "nice" (we'll leave that term undefined for now), then $[\bar{x}]$ is the indifference curve containing $\bar{x}$.

Remark: If $\succsim$ is a weak preorder, then the relation $\sim$ defined by $x' \sim x \Leftrightarrow [x' \succsim x \;\&\; x \succsim x']$ is an equivalence relation, called the equivalence relation associated with $\succsim$.

Exercise: Provide a proof that the relation $\sim$ associated with a preorder $\succsim$ is an equivalence relation.

Notice that in each of the above examples the equivalence relation "partitions" the set X into the relation's equivalence classes. Here's a formal definition of the idea of a partition:

Definition: A **partition** of a set X is a collection $\mathcal{P}$ of subsets of X that satisfies the two conditions

(1) $A, B \in \mathcal{P} \Rightarrow (A = B \text{ or } A \cap B = \emptyset)$;

(2) $\bigcup_{A \in \mathcal{P}} A = X$.

An informal statement of the above definition is that a partition of X is a collection of subsets that are *mutually exclusive* and *exhaustive.* Every member of X is in one and only one member of $\mathcal{P}$.

Theorem: If $\sim$ is an equivalence relation on a set X, then the collection of its equivalence classes is a partition of X. Conversely, if $\mathcal{P}$ is a partition of X, then the relation $\sim$ defined by

$$x \sim x' \;\Leftrightarrow\; \exists S \in \mathcal{P} : \; x, x' \in S$$

is an equivalence relation, and its equivalence classes are the elements of the partition.

Exercise: Provide a proof.

Let's apply the notions of equivalence, equivalence classes, and partitions to preferences and their utility function representations. First note that if a given preference has a utility function representation, then it has *many* utility function representations, as the following example indicates.

Example 6:
Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by $u(x) = x_1 x_2$, a Cobb-Douglas utility function, and let $\succsim$ be the preference (*i.e.*, preorder) on $\mathbb{R}^2_+$ defined as in Example 3. Obviously u is a representation of $\succsim$. Now define a new utility function $\widetilde{u}$ on $\mathbb{R}^2_+$ by $\widetilde{u}(x) = \log[u(x) + 1]$ — *i.e.*, $\widetilde{u}(x) = \log[x_1 x_2 + 1]$. It's easy to see that $\widetilde{u}$ is also a utility representation of $\succsim$.

Definition: Let $u : X \to \mathbb{R}$ be a real-valued function defined on the set X. The function $\widetilde{u} : X \to \mathbb{R}$ is an **order-preserving transformation** of u if there is a strictly increasing real function $f : u(X) \to \mathbb{R}$ such that $\widetilde{u} = f \circ u$ — *i.e.*, such that $\forall x \in X : \; \widetilde{u}(x) = f(u(x))$.

Remark: Two real-valued functions u and $\widetilde{u}$ on a set X are order-preserving transformations of one another if and only if they are both representations of the same preorder $\succsim$ on X.

Example 7:
Let X be a set and let U be the set of all real-valued functions on X. Define the relation $\sim$ on U as follows: $u \sim \widetilde{u}$ if $\widetilde{u}$ is an order-preserving transformation of u. Then $\sim$ is an equivalence relation on U. An equivalence class $[u]$ consists of all the utility functions on X that represent the same preference as u. Thus, the equivalence relation $\sim$ partitions the set U of all real-valued functions on X in such a way that each equivalence class in the partition corresponds to a *distinct* preference $\succsim$ on $\mathbb{R}^2_+$ — obviously, to a distinct *representable* preference — and each distinct representable preference corresponds to a distinct equivalence class in the partition.

Exercise: Prove that $\sim$ is an equivalence relation.

In Example 7 the set of equivalence classes plays an important role: it is the same, in every essential respect, as the set of representable preferences — every distinct preference corresponds to a distinct equivalence class. This is a common occurrence, so we assign a name and a notation to the set of equivalence classes generated by an equivalence relation:

Definition: Let $\sim$ be an equivalence relation on a set X. The **quotient** of X by $\sim$, or the **quotient set** generated by $\sim$, denoted $X/\sim$, is the set of all $\sim$-equivalence classes — *i.e.*, $X/\sim$ is the set $\{ [x] \mid x \in X \}$.

Example 6 continued:
In Example 6, $\sim$ is the indifference relation derived from the utility function u, and the quotient set $\mathbb{R}^2_+/\sim$ is the set of indifference curves for the function u and also for any order-preserving transformation of u.

Suppose someone's preference over $\mathbb{R}^2_+$ is described by a complete preorder $\succsim$. Can we represent $\succsim$ by a utility function? In other words, can we be sure that $\succsim$ is representable? Example 7 suggests that perhaps the answer is yes, that every complete preorder is representable. It turns out that this is not so. The following theorem provides conditions that guarantee that a preorder *is* representable. The theorem is followed by an example of a preference that is not representable.

Representation Theorem: If a relation R on the set $\mathbb{R}^l_+$ is complete, transitive, and continuous — *i.e.*, a complete and continuous preorder — then it is representable. Moreover, it is representable by a *continuous* utility function. (We'll define continuity for relations and functions shortly.)

Proof: Debreu, on page 56, Proposition (1), gives a proof. Jehle & Reny, on page 120, Theorem 3.1, give a proof for preorders that are complete, continuous, and strictly increasing.

Do we really need all three assumptions, or "axioms," about a preference in order to know that it is representable by a utility function? For example, it seems plausible that if we don't insist that the utility function be continuous, we may be able to at least ensure that a (possibly discontinuous) representation of R exists if we at least know that R is a complete preorder. Or perhaps if R satisfies one or more additional assumptions as well, but assumptions that are not as strong as continuity, then *that* will be enough to ensure that R is representable. What we want in this kind of situation is a collection of *counterexamples*: for each assumption in our theorem, we want an example that demonstrates that if all the remaining assumptions are satisfied, but that one isn't, then R need not be representable. One such counterexample is given below: a relation R that is complete and transitive, but is not continuous, and for which no utility function exists.

Exercise: Provide counterexamples to show that neither completeness nor transitivity can be dispensed with in the theorem above.

Example 8 (Lexicographic Preference):
This is an example of a preference relation — a complete preorder — which is not representable. Of course, it's not a *continuous* relation; otherwise we would have a counterexample to the truth of the theorem. Let $\succsim$ on $\mathbb{R}^2_+$ be defined by

$$(x_1', x_2') \succsim (x_1, x_2) \Leftrightarrow [x_1' > x_1 \text{ or } (x_1' = x_1 \ \& \ x_2' \geqq x_2)].$$

See Figure 1. How would we show that $\succsim$ is not representable? We have to show that *no* utility function could represent $\succsim$ — which is not so easy, as it turns out. The proof relies on a fairly deep mathematical result: that the set of all real numbers (or any non-degenerate real interval) is an "uncountable" set. If we accept that mathematical fact, then the proof is not so bad: we assume that $\succsim$ has a representation $u(\cdot)$, and then we use that to establish that $\mathbb{R}_+$ is countable, which we know to be false. Therefore, our assumption that $\succsim$ has a representation $u(\cdot)$ cannot be correct. This is called an *indirect proof*, or a *proof by contradiction*.

Thus, we assume that $u(\cdot)$ is a utility function for $\succsim$. For each $x \in \mathbb{R}_+$, define the two real numbers $a(x) = u(x, 1)$ and $b(x) = u(x, 2)$ (see Figure 2). Clearly, $a(x) < b(x)$ for each x, and therefore, for each x, there is a rational number $r(x)$ that lies between $a(x)$ and $b(x)$. Moreover, if $x < \widetilde{x}$, then $r(x) < b(x)$ and $b(x) < a(\widetilde{x})$ and $a(\widetilde{x}) < r(\widetilde{x})$; we therefore have $r(x) < r(\widetilde{x})$ whenever $x < \widetilde{x}$ — in particular, $x \neq \widetilde{x} \Rightarrow r(x) \neq r(\widetilde{x})$, so that $r(\cdot)$ is a *one-to-one* mapping from $\mathbb{R}_+$ to a subset of the set $\mathbb{Q}$ of rational numbers. Since any subset of $\mathbb{Q}$ is countable, this implies that $\mathbb{R}_+$ is countable, which we know is false. ■

Note that an equivalent definition of $\succsim$ is the following:

$$(x_1', x_2') \succ (x_1, x_2) \Leftrightarrow [x_1' > x_1 \text{ or } (x_1' = x_1 \ \& \ x_2' > x_2)],$$

together with

$$(x_1', x_2') \succsim (x_1, x_2) \Leftrightarrow (x_1, x_2) \not\succ (x_1', x_2').$$

Figure 1:

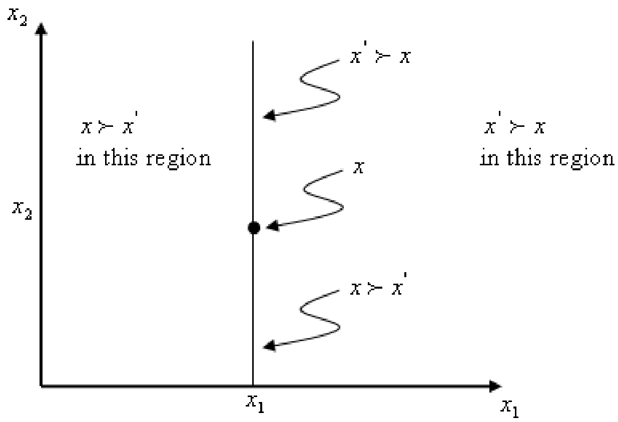

Figure 2: