Economics 519 Exam
University of Arizona
Fall 2011

Closed-book part: Don't consult books or notes until you've handed in your solutions to Problems #1, #2, and #3. After you do consult books or notes, your solutions to Problems #1, #2, and #3 will not be accepted.

1. Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$. Prove that if f is continuous and X is compact, then $f(X)$ is compact

2. Let (X, d) be a metric space and let S be the set of all Cauchy sequences in X. Define a relation $\sim$ on S as follows:

$$\{s_n\} \sim \{s'_n\} \text{ if and only if } \forall \epsilon > 0 : \exists \overline{n} \in \mathbb{N} : m, n > \overline{n} \Rightarrow d(s_m, s'_n) < \epsilon.$$

(a) Prove that $\sim$ is transitive. (The relation $\sim$ is obviously reflexive and symmetric, so you needn't prove that.)

(b) Describe the equivalence classes that comprise the partition $S/\sim$ when X is complete.

3. In the Cournot duopoly model, assume that each firm's response function r_i is continuous and maps $\mathbb{R}_+$ into an interval $Q_i = [0, b_i]$ — *i.e.*, for each firm there is an upper bound b_i beyond which the firm would not produce, no matter what quantity its rival firm produces.

(a) Use the Brouwer Fixed Point Theorem to establish that if each r_i is continuous, then a Cournot equilibrium exists.

(b) Provide a diagram to show that the assumptions here do not guarantee that the equilibrium is unique.

(c) Can you use the Banach Fixed Point Theorem here instead of the Brouwer Theorem to establish the existence of a Nash equilibrium? If so, provide a proof using the Banach Theorem; if not, explain why not.

Open-book part: For the remainder of the exam you may consult de la Fuente, Simon & Blume, and/or your notes. But once you've done so, your solutions to Problems #1, #2, and #3 will no longer be accepted.

4. Let V be a vector space, and let $\|\cdot\|_a$ and $\|\cdot\|_b$ be norms on V. For any norm $\|\cdot\|_\nu$, define *convergence w.r.t.* $\|\cdot\|_\nu$ as follows:

$$\{x_n\} \underset{\nu}{\rightarrow} \overline{x} \text{ if } \forall \epsilon > 0: \exists \overline{n} \in \mathbb{N}: n > \overline{n} \Rightarrow \|x_n - \overline{x}\|_\nu < \epsilon.$$

Prove that if the norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent, then $\{x_n\} \underset{a}{\rightarrow} \overline{x}$ if and only if $\{x_n\} \underset{b}{\rightarrow} \overline{x}$. (Prove only one of the implications in the "if and only if" statement, since the converse is proved identically.)

5. Let S denote the unit simplex in $\mathbb{R}^2$. Define the correspondence $\mu : \mathbb{R}^2 \twoheadrightarrow S$ as follows: $\mu(\mathbf{z})$ is the set of all price-lists $\mathbf{p} = (p_1, p_2)$ that maximize the value $\mathbf{p} \cdot \mathbf{z}$ of excess demand at $\mathbf{z}$; *i.e.*,

$$\mu(z_1, z_2) = \begin{cases} S, & if z_1 = z_2 \\ \{(1,0)\}, & if z_1 > z_2 \\ \{(0,1)\}, & if z_1 < z_2 \end{cases}$$

Prove that μ has a closed graph.