Fixed point theorems with applications to economics and game theory

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10.9 Remark: Approximating Maximal Elements
The set of maximal elements of a binary relation U on K is $\bigcap_{z \varepsilon K} (K \setminus U^{-1}(z))$. If U has open graph, then we may approximate this intersection by a finite intersection. This is proven in Theorem 10.11.

10.10 Definition
A set D is δ-*dense* in K if every open set of diameter δ meets D. It follows that if K is compact, then for every $\delta > 0$, K has a finite δ-dense subset.

10.11 Theorem
Let K be compact and let U be a binary relation on K with open graph. Let M be the set of maximal elements of U. For every $\varepsilon > 0$, there is a $\delta > 0$ such that if D is δ-dense in K, then $\bigcap_{z \varepsilon D} K \setminus U^{-1}(z) \subset N_\varepsilon(M)$.

10.12 Proof
Let $x \in K \setminus M$. Then there is a $y_x \in U(x)$, and since U has open graph, there is a δ_x such that $N_{\delta_x}(x) \times N_{\delta_x}(y_x) \subset Gr\ U$. Since $C = K \setminus N_\varepsilon(M)$ is compact, it is covered by a finite collection $\{N_{\delta_i}(x_i)\}$. Put $\delta = \min_i \delta_i$.

Let $x \notin N_\varepsilon(M)$. Then $x \in C$ and so $x \in N_{\delta_i}(x_i)$ for some i. Since D is δ-dense, let $z \in D \cap N_{\delta_i}(y_i)$. Since $N_{\delta_i}(x_i) \times N_{\delta_i}(y_i) \subset Gr\ U$, we have that $x \in U^{-1}(z)$ and so $x \notin K \setminus U^{-1}(z)$.

Thus $\bigcap_{z \varepsilon D} K \setminus U^{-1}(z) \subset N_\varepsilon(M)$.

CHAPTER 11

Continuity of correspondences

11.0 Remark
A correspondence is a function whose values are sets of points. Notions of continuity for correspondences can traced back to Kuratowski [1932] and Bouligand [1932]. Berge [1959, Ch. 6] and Hildenbrand [1974, Ch. B] have collected most of the relevant theorems on continuity of correspondences. It is difficult to attribute most of these theorems, but virtually all of the results of this chapter can be found in Berge [1959]. Whenever possible, citations are provided for theorems not found there. Due to slight differences in terminology, the proofs presented here are generally not identical to those of Berge. A particular difference in terminology is that Berge requires compact-valuedness as part of the definition of upper semi-continuity. Since these properties seem to be quite distinct, that requirement is not made here. In applications, it frequently makes no difference, as the correspondences under consideration have compact values anyway. Moore [1968] has catalogued a number of differences between different possible definitions of semi-continuity. The term hemi-continuity has now replaced semi-continuity in referring to correspondences. It helps to avoid confusion with semi-continuity of real-valued functions.

The chief use of correspondences in economic and game theoretic problems is the linking up of multi-player situations and single-player situations. For example, the problem of finding a maximal element of a binary relation as discussed in Chapter 7 is a single-player problem. The solution to the problem does not depend on the actions of others. As another example, the problem of finding an equilibrium price vector can be reduced to a single-player maximization problem as is shown in Chapter 8. The problem of finding a noncooperative equilibrium of a multi-player game is on the face of it of a different sort. It amounts to solving several interdependent individual maximization problems simultaneously. Given a choice of variables for all but one of the maximization problems we can find the set of solutions for the remaining problem. This solution will in general depend on the choices of the other players and so defines a correspondence mapping the set of joint choice variables into itself. A noncooperative equilibrium will be a fixed point of this correspondence. Theorems on the existence of fixed points for correspondences are presented in Chapter 15. There are of course other uses for correspondences, even in single-player problems such as the equilibrium price problem, as is shown in Chapter 18. On the other hand, it is also possible to reduce multi-player situations to situations involving a single fictitious player, as in 19.7.

The general method of proof for results about correspondences is to reduce the problem to one involving (single-valued) functions. The single-valued function will either approximate the correspondence or be a selection from it. The theorems of Chapters 13 and 14 are all in this vein. In a sense these techniques eliminate the need for any othe theorems about correspondences, since they can be proved by using only theorems about functions. Thus it is always possible to substitute the use of Brouwer's fixed point theorem for the use of Kakutani's fixed point theorem, for example. While Brouwer's theorem is marginally easier to prove, it is frequently the case that it is more intuitive to define a correspondence than to construct an approximating function.

11.1 Definition
Let 2^Y denote the power set of Y, i.e., the collection of all subsets of Y. A *correspondence* (or *multivalent function*) γ from X to Y is a function from X to the family of subsets of Y. We denote this by $\gamma : X \twoheadrightarrow Y$. (Binary relations as defined in 7.1 can be viewed as correspondences from a set into itself.) For a correspondence $\gamma : E \twoheadrightarrow F$, let $Gr\ \gamma$ denote the *graph* of γ, i.e.,

$$Gr\ \gamma = \{(x,y) \in E \times F : y \in \gamma(x)\}.$$

Likewise, for a function $f : E \to F$

$$Gr\ f = \{(x,y) \in E \times F : y = f(x)\}.$$

11.2 Definition
Let $\gamma : X \twoheadrightarrow Y$, $E \subset Y$ and $F \subset X$. The *image* of F under γ is defined by

$$\gamma(F) = \bigcup_{x \in F} \gamma(x).$$

The *upper* (or *strong*) *inverse* of E under γ, denoted $\gamma^+[E]$, is defined by

$$\gamma^+[E] = \{x \in X : \gamma(x) \subset E\}.$$

The *lower* (or *weak*) *inverse* of E under γ, denoted $\gamma^-[E]$, is defined by

$$\gamma^-[E] = \{x \in X : \gamma(x) \cap E \neq \varnothing\}.$$

For $y \in Y$, set

$$\gamma^{-1}(y) = \{x \in X : y \in \gamma(x)\}.$$

Note that $\gamma^{-1}(y) = \gamma^-[\{y\}]$. (If U is a binary relation on X, i.e., $U : X \twoheadrightarrow X$, then this definition is consistent with the definition of $U^{-1}(y)$ in 7.1.)

11.3 Definition
A correspondence $\gamma : X \twoheadrightarrow Y$ is called *upper hemi-continuous (uhc) at* x if whenever x is in the upper inverse of an open set so is a neighborhood of x; and γ is *lower hemi-continuous (lhc) at* x if whenever x is in the lower inverse of an open set so is a neighborhood of x. The correspondence $\gamma : X \twoheadrightarrow Y$ is *upper hemi-continuous* (resp. *lower hemi-continuous*) if it is upper hemi-continuous (resp. lower hemi-continuous) at every $x \in X$. Thus γ is upper hemi-continuous (resp. lower hemi-continuous) if the upper (resp. lower) inverses of open sets are open. A correspondence is called *continuous* if it is both upper and lower hemi-continuous.

11.4 Note
If $\gamma : X \twoheadrightarrow Y$ is singleton-valued it can be considered as a function from X to Y and we may sometimes identify the two. In this case the upper and lower inverses of a set coincide and agree with the inverse regarded as a function. Either form of hemi-continuity is equivalent to continuity as a function. The term "semi-continuity" has been used to mean hemi-continuity, but this usage can lead to confusion when discussing real-valued singleton correspondences. A semi-continuous real-valued function (2.27) is not a hemi-continuous correspondence unless it is also continuous.

11.5 Definition
The correspondence $\gamma : E \twoheadrightarrow F$ is said to be *closed at* x if whenever $x^n \to x$, $y^n \in \gamma(x^n)$ and $y^n \to y$, then $y \in \gamma(x)$. A correspondence is said to be *closed* if it is closed at every point of its domain, i.e., if its graph is closed. The correspondence γ is said to be *open* or have *open graph* if $Gr\ \gamma$ is open in $E \times F$.

11.6 Definition
A correspondence $\gamma : E \twoheadrightarrow F$ is said to have *open* (resp. *closed*) *sections* if for each $x \in E$, $\gamma(x)$ is open (resp. closed) in F, and for each $y \in F$, $\gamma^-[\{y\}]$ is open (resp. closed) in E.

11.7 Note

There has been some blurring in the literature of the distinction between closed correspondences and upper hemi-continuous correspondences. The relationship between the two notions is set forth in 11.8 and 11.9 below. For closed-valued correspondences into a compact space the two definitions coincide and the distinction may seem pedantic. Nevertheless the distinction is important in some circumstances. (See, for example, 11.23 below or Moore [1968].)

11.8 Examples: Closedness vs. Upper Hemi-continuity

In general, a correspondence may be closed without being upper hemi-continuous, and vice versa.

Define $\gamma : \mathbf{R} \twoheadrightarrow \mathbf{R}$ via

$$\gamma(x) = \begin{cases} \{1/x\} & \text{for } x \neq 0 \\ \{0\} & \text{for } x = 0 \end{cases}$$

Then γ is closed but not upper hemi-continuous.

Define $\mu : \mathbf{R} \twoheadrightarrow \mathbf{R}$ via $\mu(x) = (0,1)$. Then μ is upper hemi-continuous but not closed.

11.9 Proposition: Closedness, Openness and Hemi-continuity

Let $E \subset \mathbf{R}^m$, $F \subset \mathbf{R}^k$ and let $\gamma : E \twoheadrightarrow F$.

(a) If γ is upper hemi-continuous and closed-valued, then γ is closed.

(b) If F is compact and γ is closed, then γ is upper hemi-continuous.

(c) If γ is open, then γ is lower hemi-continuous.

(d) If γ is singleton-valued at x and upper hemi-continuous at x, then γ is continuous at x.

(e) If γ has open lower sections, then γ is lower hemi-continuous.

11.10 Proof

(a) Suppose $(x,y) \notin Gr\ \gamma$. Then since γ is closed-valued, there is a closed neighborhood U of y disjoint from $\gamma(x)$. Then $V = U^c$ is an open neighborhood of $\gamma(x)$. Since γ is upper hemi-continuous, $\gamma^+[V]$ contains an open neighborhood W of x, i.e., $\gamma(z) \subset V$ for all $z \in W$. Thus $(W \times U) \cap Gr\ \gamma = \varnothing$ and $(x,y) \in W \times U$. Hence the complement of $Gr\ \gamma$ is open, so $Gr\ \gamma$ is closed.

(b) Suppose not. Then there is some x and an open neighborhood U of $\gamma(x)$ such that for every neighborhood V of x, there is a $z \in V$ with $\gamma(z) \not\subset U$. Thus we can find $z^n \to x$, $y^n \in \gamma(z^n)$ with $y^n \notin U$. Since F is compact, $\{y^n\}$ has a convergent subsequence converging to $y \notin U$. But since γ is closed, $(x,y) \in Gr\ \gamma$, so $y \in \gamma(x) \subset U$, a contradiction.

(c) Exercise.

(d) Exercise.

(e) Exercise.

11.11 Proposition: Sequential Characterizations of Hemi-continuity

Let $E \subset \mathbf{R}^m$, $F \subset \mathbf{R}^k$, $\gamma : E \twoheadrightarrow F$.

(a) If γ is compact-valued, then γ is upper hemi-continuous at x if and only if for every sequence $x^n \to x$ and $y^n \in \gamma(x^n)$ there is a convergent subsequence of $\{y^n\}$ with limit in $\gamma(x)$.

(b) Then γ is lower hemi-continuous if and only if $x^n \to x$ and $y \in \gamma(x)$ imply that there is a sequence $y^n \in \gamma(x^n)$ with $y^n \to y$.

11.12 Proof

(a) Suppose γ is upper hemi-continuous at x, $x^n \to x$ and $y^n \in \gamma(x^n)$. Since γ is compact-valued, $\gamma(x)$ has a bounded neighborhood U. Since γ is upper hemi-continuous, there is a neighborhood V of x such that $\gamma(V) \subset U$. Thus $\{y^n\}$ is eventually in U, thus bounded, and so has a convergent subsequence. Since compact sets are closed, this limit belongs to $\gamma(x)$.

Now suppose that for every sequence $x^n \to x$, $y^n \in \gamma(x^n)$, there is a subsequence of $\{y^n\}$ with limit in $\gamma(x)$. Suppose γ is not upper hemi-continuous; then there is a neighborhood U of x and a sequence $z^n \to x$ with $y^n \in \gamma(z^n)$ and $y^n \notin U$. Such a sequence $\{y^n\}$ can have no subsequence with limit in $\gamma(x)$, a contradiction.

(b) Exercise.

11.13 Definition

A convex set F is a *polytope* if it is the convex hull of a finite set. In particular, a simplex is a polytope.

11.14 Proposition: Open Sections vs. Open Graph (cf. Shafer [1974], Bergstrom, Parks, and Rader [1976])

Let $E \subset \mathbf{R}^m$ and $F \subset \mathbf{R}^k$ and let F be a polytope. If $\gamma : E \twoheadrightarrow F$ is convex-valued and has open sections, then γ has open graph.

11.15 Proof

Let $y \in \gamma(x)$. Since γ has open sections and F is a polytope, there is a polytope neighborhood U of y contained in $\gamma(x)$. Let $U = co\ \{y^1,\ldots,y^n\}$. Since γ has open sections, for each i there is a neighborhood of x, V_i, such that $y^i \in \gamma(z)$ for all $z \in V_i$. Put

$V = \bigcap_{i=1}^{n} V_i$ and $W = V \times U$ and let $(x',y') \in W$. Then $y^i \in \gamma(x')$, $i = 1,...,n$ and $y' \in U = co\ \{y^1,...,y^n\} \subset co\ \gamma(x')$, since γ is convex-valued. Thus W is a neighborhood of (x,y) completely contained in $Gr\ \gamma$.

11.16 Proposition: Upper Hemi-continuous Image of a Compact Set
Let $\gamma : E \twoheadrightarrow F$ be upper hemi-continuous and compact-valued and let $K \subset E$ be compact. Then $\gamma(K)$ is compact.

11.17 Proof (Berge [1959])
Let $\{U_\alpha\}$ be an open covering of $\gamma(K)$. Since $\gamma(x)$ is compact, there is a finite subcover $U_{x^1},...,U_{x^{n_x}}$, of $\gamma(x)$. Put $V_x = U_x^1 \cup , \ldots , \cup U_{x^{n_x}}$. Then since γ is upper hemi-continuous, $\gamma^+[V_x]$ is open and contains x. Hence K is covered by a finite number of $\gamma^+[V_x]$'s and the corresponding U_x^i's are a finite cover of $\gamma(K)$.

11.18 Exercise: Miscellaneous Facts about Hemi-continuous Correspondences
Let $E \subset \mathbf{R}^m$.

(a) Let $\gamma : E \twoheadrightarrow \mathbf{R}^m$ be upper hemi-continuous with closed values. Then the set of fixed points of γ, i.e., $\{x \in E : x \in \gamma(x)\}$, is a closed (possibly empty) subset of E.

(b) Let $\gamma,\mu : E \twoheadrightarrow \mathbf{R}^m$ be upper hemi-continuous with closed values. Then $\{x \in E : \mu(x) \cap \gamma(x) \neq \varnothing\}$ is a closed (possibly empty) subset of E.

(c) Let $\gamma : E \twoheadrightarrow \mathbf{R}^m$ be lower hemi-continuous. Then $\{x \in E : \gamma(x) \neq \varnothing\}$ is an open subset of E.

(d) Let $\gamma : E \twoheadrightarrow \mathbf{R}^m$ be upper hemi-continuous. Then $\{x \in E : \gamma(x) \neq \varnothing\}$ is a closed subset of E.

(e) Let $X \subset \mathbf{R}^m$ be closed, convex, and bounded below and let $\beta : \mathbf{R}_+^{m+1} \twoheadrightarrow X$ be defined by $\beta(p,M) = \{x \in X : p \cdot x \leqslant M\}$, where $M \in \mathbf{R}_+$ and $p \in \mathbf{R}_+^m$. In other words, β is a budget correspondence for the consumption set X. Show that β is upper hemi-continuous; and if there is some $x \in X$ satisfying $p \cdot x < M$, then β is lower hemi-continuous at (p,M).

11.19 Proposition: Closure of a Correspondence
Let $E \subset \mathbf{R}^m$ and $F \subset \mathbf{R}^k$

(a) Let $\gamma : E \twoheadrightarrow F$ be upper hemi-continuous at x. Then $\bar{\gamma} : E \twoheadrightarrow F$, defined by

$$\bar{\gamma}(x) = \textit{closure (in } F\textit{) of } \gamma(x)$$

is upper hemi-continuous at x.

(b) The converse of (a) is not true.

(c) The correspondence $\gamma : E \twoheadrightarrow F$ is lower hemi-continuous at x if and only if $\overline{\gamma} : E \twoheadrightarrow F$ is lower hemi-continuous at x.

11.20 Proof

Exercise. Hints:

(a) Use the fact that if E and F are disjoint closed sets in $\mathbf{R}^m$, then they have disjoint open neighborhoods.

(b) Consider $\gamma : \mathbf{R} \twoheadrightarrow \mathbf{R}$ via $\gamma(x) = \{x\}^c$.

(c) Use the Cantor diagonal process and 11.11.

11.21 Proposition: Intersections of Correspondences

Let $E \subset \mathbf{R}^m$, $F \subset \mathbf{R}^k$ and $\gamma,\mu : E \twoheadrightarrow F$, and define $(\gamma \cap \mu) : E \twoheadrightarrow F$ by $(\gamma \cap \mu)(x) = \gamma(x) \cap \mu(x)$. Suppose $\gamma(x) \cap \mu(x) \neq \varnothing$.

(a) If γ and μ are upper hemi-continuous at x and closed-valued, then $(\gamma \cap \mu)$ is upper hemi-continuous at x. (Hildenbrand [1974, Prop. 2a., p. 23].)

(b) If μ is closed at x and γ is upper hemi-continuous at x and $\gamma(x)$ is compact then $(\gamma \cap \mu)$ is upper hemi-continuous at x. (Berge [1959, Th. 7, p. 117].)

(c) If γ is lower hemi-continuous at x and if μ has open graph, then $(\gamma \cap \mu)$ is lower hemi-continuous at x. (Prabhakar and Yannelis [1983, Lemma 3.2].)

11.22 Proof

Let U be an open neighborhood of $\gamma(x) \cap \mu(x)$. Put $C = \gamma(x) \cap U^c$.

(a) Note that C is closed and $\mu(x) \cap C = \varnothing$. Thus there are disjoint open sets V_1 and V_2 with $\mu(x) \subset V_1$, $C \subset V_2$. Since μ is upper hemi-continuous at x, there is a neighborhood W_1 of x with $\mu(W_1) \subset V_1 \subset V_2^c$. Now $\gamma(x) \subset U \cup V_2$, which is open and so x has a neighborhood W_2 with $\gamma(W_2) \subset U \cup V_2$, as γ is upper hemi-continuous at x. Put $W = W_1 \cap W_2$. Then for $z \in W$, $\gamma(z) \cap \mu(z) \subset V_2^c \cap (U \cup V_2) \subset U$. Thus $(\gamma \cap \mu)$ is upper hemi-continuous at x.

(b) Note that in this case C is compact and $\mu(x) \cap C = \varnothing$. Since μ is closed at x, if $y \notin \mu(x)$ then we cannot have $y^n \to y$, where $y^n \in \mu(x^n)$ and $x^n \to x$. Thus there is a neighborhood U_y of y and W_y of x with $\mu(W_y) \subset U_y^c$. Since C is compact, we can write $C \subset V_2 = U_{y^1} \cup \cdots \cup U_{y^n}$; so setting $W_1 = W_{y^1} \cap \cdots \cap W_{y^n}$, we have $\mu(W_1) \subset V_2^c$. The rest of the proof is as in (a).

(c) Let U be open and let $y \in (\gamma \cap \mu)(x) \cap U$. Since μ has open graph, there is a neighborhood $W \times V$ of (x,y)

contained in $Gr\ \mu$. Since γ is lower hemi-continuous, $\gamma^{-}[U \cap V] \cap W$ is a neighborhood of x, and if $z \in \gamma^{-}[U \cap V] \cap W$, then $y \in (\gamma \cap \mu)(z) \cap U$. Thus $(\gamma \cap \mu)$ is lower hemi-continuous.

11.23 Proposition: Composition of Correspondences
Let $\mu : E \twoheadrightarrow F$, $\gamma : F \twoheadrightarrow G$. Define $\gamma \circ \mu : E \twoheadrightarrow G$ via $\gamma \circ \mu(x) = \bigcup_{y \in \mu(x)} \gamma(y)$.

(a) If γ and μ are upper hemi-continuous, so is $\gamma \circ \mu$.
(b) If γ and μ are lower hemi-continuous, so is $\gamma \circ \mu$.
(c) If γ and μ are closed, $\gamma \circ \mu$ may fail to be closed.

11.24 Proof
Exercise. Hint for (c) (Moore [1968]): Let $E = \{\alpha \in \mathbf{R} : -\frac{\pi}{4} \leqslant \alpha \leqslant \frac{\pi}{4}\}$, $F = \{(x_1,x_2) \in \mathbf{R}^2 : x_1 \geqslant 0\}$ and $G = \mathbf{R}$. Set $\mu(\alpha) = \{(x_1,x_2) \in F : |x_2| \leqslant |x_1 \tan \alpha|;\ \alpha x_2 \geqslant 0\}$, i.e., $\mu(\alpha)$ is the set of points in F lying between the x_1-axis and a ray making angle α with the axis. Set $\gamma((x_1,x_2)) = \{x_2\}$.

11.25 Proposition: Products of Correspondences
Let $\gamma_i : E \twoheadrightarrow F_i$, $i = 1,\ldots,k$.

(a) If each γ_i is upper hemi-continuous at x and compact-valued, then

$$\prod_i \gamma_i : z \mapsto \prod_i \gamma_i(z)$$

is upper hemi-continuous at x and compact-valued.

(b) If each γ_i is lower hemi-continuous at x, then $\prod_i \gamma_i$ is lower hemi-continuous at x.
(c) If each γ_i is closed at x, then $\prod_i \gamma_i$ is closed at x.
(d) If each γ_i has open graph, then $\prod_i \gamma_i$ has open graph.

11.26 Proof
Exercise. Assertion (a) follows from 11.11(a), (b) from 11.11(b) and (c) and (d) from the definitions.

11.27 Proposition: Sums of Correspondences
Let $\gamma_i : E \twoheadrightarrow F_i$, $i = 1,\ldots,k$.

(a) If each γ_i is upper hemi-continuous at x and compact-valued, then

$$\sum_i \gamma_i : z \mapsto \sum_i \gamma_i(z)$$

is upper hemi-continuous at x and compact-valued.

(b) If each γ_i is lower hemi-continuous at x, then $\sum_i \gamma_i$ is lower hemi-continuous at x.
(c) If each γ_i has open graph, then $\sum_i \gamma_i$ has open graph.

11.28 Proof
Exercise. Assertion (a) follows from 2.43 and 11.11(a), (b) from 11.11(b), and (c) from the definitions.

11.29 Proposition: Convex Hull of a Correspondence
Let $\gamma : E \twoheadrightarrow F$, where F is convex.
(a) If γ is compact-valued and upper hemi-continuous at x, then

$$co\ \gamma : z \mapsto\!\!\!\rightarrow co\ \gamma(z)$$

is upper hemi-continuous at x.
(b) If γ is lower hemi-continuous at x, $co\ \gamma$ is lower hemi-continuous at x.
(c) If γ has open graph, then $co\ \gamma$ has open graph.
(d) Even if γ is a compact-valued closed correspondence, $co\ \gamma$ may still fail to be closed.

11.30 Proof
The proof is left as an exercise. For parts (a) and (b) use Caratheodory's theorem (2.3) and 11.9(c) and 11.11. For part (d) consider the correspondence $\gamma : \mathbf{R} \twoheadrightarrow \mathbf{R}$ via

$$\gamma(x) = \begin{cases} \{0,\ 1/x\} & x \neq 0 \\ \{0\} & x = 0. \end{cases}$$

11.31 Proposition: Open Sections vs. Open Graph Revisited
Let $E \subset \mathbf{R}^m$ and $F \subset \mathbf{R}^k$ and let F be a polytope. If $\gamma : E \twoheadrightarrow F$ has open sections, then $co\ \gamma$ has open graph.

11.32 Proof
By 11.14, we need only show that $co\ \gamma$ has open sections. Since $\gamma(x)$ is open for each x, so is $co\ \gamma(x)$. (Exercise 2.5c.) Next let $x \in (co\ \gamma)^{-}[\{y\}]$, i.e., $y \in co\ \gamma(x)$. We wish to find a neighborhood U of x such that $w \in U$ implies $y \in co\ \gamma(w)$. Since $y \in co\ \gamma(x)$, we can write $y = \sum_{i=1}^{n} \lambda_i z_i$, where each $z_i \in \gamma(x)$ and the λ_i's are nonnegative and sum to unity. Since γ has open sections, for each i there is a neighborhood U_i of x in $\gamma^{-}[\{z_i\}]$. Setting $U = \bigcap_{i=1}^{n} U_i$, we have that $w \in U$ implies $z_i \in \gamma(w)$ for all i, so that $y \in co\ \{z_1,\ldots,z_n\} \subset co\ \gamma(w)$. Thus $co\ \gamma$ has open sections.

11.33 Note
It follows from 11.29(d) that the analogue of Proposition 11.31 for correspondences with closed sections is false.

The maximum theorem

12.0 Remarks
One of the most useful and powerful theorems employed in mathematical economics and game theory is the "maximum theorem." It states that the set of solutions to a maximization problem varies upper hemi-continuously as the constraint set of the problem varies in a continuous way. Theorem 12.1 is due to Berge [1959] and considers the case of maximizing a continuous real-valued function over a compact set which varies continuously with some parameter vector. The set of solutions is an upper hemi-continuous correspondence with compact values. Furthermore, the value of the maximized function varies continuously with the parameters. Theorem 12.3 is due to Walker [1979] and extends Berge's theorem to the case of maximal elements of an open binary relation. Theorem 12.3 allows the binary relation as well as the constraint set to vary with the parameters. Similar results may be found in Sonnenschein [1971] and Debreu [1969]. Theorem 12.5 weakens the requirement of open graph to the requirement that the nonmaximal set be open, at the expense of requiring the constraint set to fixed and independent of the parameters. The remaining theorems are applications of the principles to problems encountered in later chapters.

In the statement of the theorems, the set G should be interpreted as the set of parameters, and Y or X as the set of alternatives. For instance, in 11.8(e) it is shown that the budget correspondence, $\beta : (p,m) \mapsto\!\!\to \{x \in \mathbf{R}^m_+ : p \cdot x \leqslant m, x \geqslant 0\}$ is continuous for $m > 0$ and compact-valued for $p > 0$. The set of parameters is then $G = \mathbf{R}^m_{++} \times \mathbf{R}_{++}$, the set of price-income pairs. If a consumer has a preference relation satisfying the hypotheses of 7.5, then Theorem 12.3 says that his demand correspondence is upper hemi-continuous. Likewise, supply correspondences are upper hemi-continuous, so that excess demand correspondences are upper hemi-continuous, provided consumers have strictly positive income.

12.1 Theorem (Berge [1959])
Let $G \subset \mathbf{R}^m$, $Y \subset \mathbf{R}^k$ and let $\gamma : G \twoheadrightarrow Y$ be a compact-valued correspondence. Let $f : Y \to \mathbf{R}$ be continuous. Define $\mu : G \twoheadrightarrow Y$ by $\mu(x) = \{y \in \gamma(x) : y \text{ maximizes } f \text{ on } \gamma(x)\}$, and $F : G \to \mathbf{R}$ by $F(x) = f(y)$ for $y \in \mu(x)$. If γ is continuous at x, then μ is closed and upper hemi-continuous at x and F is continuous at x. Furthermore, μ is compact-valued.

12.2 Proof
First note that since γ is compact-valued, μ is nonempty and compact-valued. It suffices to show that μ is closed at x, for then $\mu = \gamma \cap \mu$ and 11.21(b) implies that μ is upper hemi-continuous at x. Let $x^n \to x$, $y^n \in \mu(x^n)$, $y^n \to y$. We wish to show $y \in \mu(x)$ and $F(x^n) \to F(x)$. Since γ is upper hemi-continuous and compact-valued, 11.9(a) implies that indeed $y \in \gamma(x)$. Suppose $y \notin \mu(x)$. Then there is $z \in \gamma(x)$ with $f(z) > f(y)$. Since γ is lower hemi-continuous at x, by 11.11 there is a sequence $z^n \to z$, $z^n \in \gamma(x^n)$. Since $z^n \to z$, $y^n \to y$ and $f(z) > f(y)$, the continuity of f implies that eventually $f(z^n) > f(y^n)$, contradicting $y^n \in \mu(x^n)$. Now $F(x^n) = f(y^n) \to f(y) = F(x)$, so F is continuous at x.

12.3 Theorem (Walker [1979], cf. Sonnenschein [1971])
Let $G \subset \mathbf{R}^m$, $Y \subset \mathbf{R}^k$, and let $\gamma : G \twoheadrightarrow Y$ be upper hemi-continuous with compact values. Let $U : Y \times G \twoheadrightarrow Y$ have an open graph. Define $\mu : G \twoheadrightarrow Y$ by $\mu(x) = \{y \in \gamma(x) : U(y,x) \cap \gamma(x) = \varnothing\}$. If γ is closed and lower hemi-continuous at x, then μ is closed at x. If in addition, γ is upper hemi-continuous at x, then μ is upper hemi-continuous at x. Further, μ has compact (but possibly empty) values.

12.4 Proof
Since U has open graph, $\mu(x)$ is closed (its complement being clearly open) in $\gamma(x)$, which is compact. Thus μ has compact values.

Let $x^n \to x$, $y^n \in \mu(x^n)$, $y^n \to y$. We wish to show that $y \in \mu(x)$. Since γ is closed and $y^n \in \mu(x^n) \subset \gamma(x^n)$, $y \in \gamma(x)$. Suppose $y \notin \mu(x)$. Then there exists $z \in \gamma(x)$ with $z \in U(y,x)$. Since γ is lower hemi-continuous at x, by 11.11 there is a sequence $z^n \to z$, $z^n \in \gamma(x^n)$. Since U has open graph, $z^n \in U(y^n,x^n)$ eventually, which contradicts $y^n \in \mu(x^n)$. Thus μ is closed at x.

If γ is upper hemi-continuous as well since $\mu = \mu \cap \gamma$, and μ is closed at x, 11.21(b) implies that μ is upper hemi-continuous at x.

12.5 Proposition
Let $G \subset \mathbf{R}^m$, $Y \subset \mathbf{R}^k$ and let $U : G \times Y \twoheadrightarrow Y$ satisfy the following condition.

If $z \in U(y,x)$, then there is $z' \in U(y,x)$ such that $(y,x) \in int\ U^{-}[\{z'\}]$.

Define $\mu(x) = \{y \in Y : U(y,x) = \varnothing\}$. Then μ is closed.

12.6 Proof
Let $x^n \to x$, $y^n \in \mu(x^n)$, $y^n \to y$. Suppose $y \notin \mu(x)$. Then there must be $z \in U(y,x)$ and so by hypothesis there is some z' such that $(y,x) \in int\ U^{-}[\{z'\}]$. But then for n large enough, $z' \in U(y^n,x^n)$, which contradicts $y^n \in \mu(x^n)$.

12.7 Theorem (cf. Theorem 22.2, Walker [1979], Green [1984])
Let $X_i \subset \mathbf{R}^{k_i}$, $i = 1,\dots,n$ be compact and put $X = \prod_{i=1}^{n} X_i$. Let $G \subset \mathbf{R}^k$ and for each i, let $S_i : X \times G \twoheadrightarrow X_i$ be continuous with compact values and $U_i : X \times G \twoheadrightarrow X_i$ have open graph. Define $E : G \twoheadrightarrow X$ via

$$E(g) = \{x \in X : \text{for each } i,\ x_i \in S_i(x,g);\ U_i(x,g) \cap S_i(x,g) = \varnothing\}.$$

Then E has compact values, is closed and upper hemi-continuous.

12.8 Proof
By 11.9 it suffices to prove that E is closed, so suppose that $(g,x) \notin Gr\ E$. Then for some i, either $x_i \notin S_i(x,g)$ or $U_i(x,g) \cap S_i(x,g) \neq \varnothing$. By 11.9, S_i is closed and so in the first case a neighborhood of (x,g) is disjoint from $Gr\ E$. In the second case, let $z_i \in U_i(x,g) \cap S_i(x,g)$. Since U_i has open graph, there are neighborhoods V of z_i and W_1 of (x,g) such that $W \times V \subset Gr\ U_i$. Since S_i is lower hemi-continuous, there is a neighborhood W_2 of (x,g) such that $(x',g') \in W_2$ implies $S_i(x',g') \cap V \neq \varnothing$. Thus $W_1 \cap W_2$ is a neighborhood of (x,g) disjoint from $Gr\ E$. Thus $Gr\ E$ is closed.

12.9 Proposition
Let $K \subset \mathbf{R}^m$ be compact, $G \subset \mathbf{R}^k$, and let $\gamma : K \times G \twoheadrightarrow K$ be closed. Put $F(g) = \{x \in K : x \in \gamma(x,g)\}$. Then $F : G \twoheadrightarrow K$ has compact values, is closed and upper hemi-continuous.

12.10 Proof
It suffices to prove that F is closed, but this is immediate.

12.11 Proposition
Let $K \subset \mathbf{R}^m$ be compact, $G \subset \mathbf{R}^k$, and let $\gamma : K \times G \twoheadrightarrow \mathbf{R}^m$ be upper hemi-continuous and have compact values. Put $Z(g) = \{x \in K : 0 \in \gamma(x,g)\}$. Then $Z : G \twoheadrightarrow K$ has compact values, is closed and upper hemi-continuous.

12.12 Proof
Exercise.

CHAPTER 13

Approximation of correspondences

13.0 Remark
In Theorem 13.3 we show that we can approximate the graph of a nonempty and convex-valued closed correspondence by the graph of a continuous function, in the sense that for any $\varepsilon > 0$ the graph of the continuous function can be chosen to lie in an ε-neighborhood of the graph of the correspondence. This result is due to von Neumann [1937] and is fundamental in extending the earlier results for functions to correspondences.

13.1 Lemma (Cellina [1969])
Let $\gamma : E \twoheadrightarrow F$ be upper hemi-continuous and have nonempty compact convex values, where $E \subset \mathbf{R}^m$ is compact and $F \subset \mathbf{R}^k$ is convex. For $\delta > 0$ define γ^δ via $\gamma^\delta(x) = co \bigcup_{z \in N_\delta(x)} \gamma(z)$. Then for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$Gr\ \gamma^\delta \subset N_\varepsilon(Gr\ \gamma).$$

(Note that this does *not* say that $\gamma^\delta(x) \subset N_\varepsilon(\gamma(x))$ for all x.)

13.2 Proof
Suppose not. Then we must have a sequence (x^n,y^n) with $(x^n,y^n) \in Gr\ \gamma^{(\frac{1}{n})}$ such that $dist\ ((x^n,y^n), Gr\ \gamma) \geqslant \varepsilon > 0$. Now $(x^n,y^n) \in Gr\ \gamma^{(\frac{1}{n})}$ means

$$y^n \in \gamma^{(\frac{1}{n})}(x^n), \text{ so } y^n \in co \bigcup_{z \in N_{(\frac{1}{n})}(x^n)} \gamma(z).$$

By Caratheodory's theorem there exist

$$y^{0,n},\ldots,y^{k,n} \in \bigcup_{z \in N_{(\frac{1}{n})}(x^n)} \gamma(z)$$

such that $y^n = \sum_{i=0}^{k} \lambda_i^n y^{i,n}$ with $\lambda^i \geqslant 0$, $\sum \lambda^i = 1$, and $y^{i,n} \in \gamma(z^{i,n})$ with