ECON 519 MIDTERM EXAM
FALL 2016
SOLUTIONS

(1)

(a) $\forall x \in \mathbb{R}: (f+g)(x) = f(x) + g(x)$

$\forall x \in \mathbb{R}, \forall \lambda \in \mathbb{R}: (\lambda f)(x) = \lambda f(x).$

(b) LET $f_1, f_2 \in \mathcal{P}$: (NOTE THAT $\mathcal{P}$ IS NONEMPTY)

$f_1(x) = a_0 + a_1 x + \cdots + a_n x^n$

$f_2(x) = b_0 + b_1 x + \cdots + b_m x^m$; WLOG ASSUME $m \leq n$.

DEFINE $c_i = a_i + b_i \quad (i = 1, \ldots, m)$

AND IF $m < n$, DEFINE $c_i = a_i \quad (i = m+1, \ldots, n)$.

THEN $(f_1 + f_2)(x) = c_0 + c_1 x + \cdots + c_n x^n$, $\forall x \in \mathbb{R}$;

i.e., $f_1 + f_2 \in \mathcal{P}$.

LET $f \in \mathcal{P}$ AND $\lambda \in \mathbb{R}$:

$f(x) = a_0 + a_1 x + \cdots + a_n x^n$.

DEFINE $c_i = \lambda a_i \quad (i = 1, \ldots, n)$.

THEN
$$
\begin{aligned}
(\lambda f)(x) &= \lambda f(x) \\
&= \lambda (a_0 + a_1 x + \cdots + a_n x^n) \\
&= c_0 + c_1 x + \cdots + c_n x^n,
\end{aligned}
$$

i.e., $\lambda f \in \mathcal{P}$.

WE KNOW THAT $\mathcal{P} \subseteq \mathcal{F}$ & $\mathcal{P} \neq \emptyset$ AND IF $f_1, f_2 \in \mathcal{P} \Rightarrow f_1 + f_2 \in \mathcal{P}$

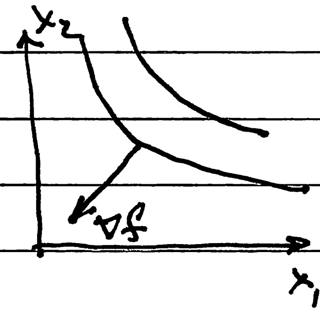
AND $\lambda \in \mathbb{R}, f \in \mathcal{P} \Rightarrow \lambda f \in \mathcal{P}$,

THEN $\mathcal{P}$ IS A VECTOR SUBSPACE OF $\mathcal{F}$.

(2) $f : \mathbb{R}^2_{++} \to \mathbb{R}$

$$f(x_1, x_2) = \frac{1}{x_1} + \frac{1}{x_2} = x_1^{-1} + x_2^{-1}$$

$$f_1 = -x_1^{-2} = -\frac{1}{x_1^2}, \quad f_{22} = -x_2^{-2} = -\frac{1}{x_2^2}$$



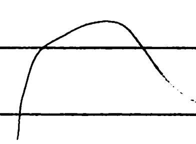
$$\frac{f_1}{f_2} = \frac{1/x_1^2}{1/x_2^2} = \frac{x_2^2}{x_1^2} = \left(\frac{x_2}{x_1}\right)^2$$

$$\nabla f = \left(-\frac{1}{x_1^2}, -\frac{1}{x_2^2}\right)$$

$$f_{11} = 2x_1^{-3}, \quad f_{22} = 2x_2^{-3}, \quad f_{12} = f_{21} = 0$$

$$H = \begin{bmatrix} 2x_1^{-3} & 0 \\ 0 & 2x_2^{-3} \end{bmatrix} = \begin{bmatrix} \frac{2}{x_1^3} & 0 \\ 0 & \frac{2}{x_2^3} \end{bmatrix}$$

$$\left.\begin{aligned} |H| = 4x_1^{-3}x_2^{-3} = \frac{4}{(x_1x_2)^3} > 0 \\ f_{11} > 0, \; f_{22} > 0 \end{aligned}\right\} \therefore f \text{ IS CONVEX}$$



(3) LET S BE A CONVEX SUBSET OF $\mathbb{R}^n$; LET $f: S \to \mathbb{R}$; LET $a, b, c \in \mathbb{R}$ WITH $a > 0$; AND $\forall x \in S$ LET $g(x) = af(x) + b$. THEN f IS CONCAVE IF AND ONLY IF g IS CONCAVE.

PROOF:

ASSUME THAT f IS CONCAVE, AND LET $x, x' \in S$ AND $\lambda \in (0,1)$. THEN

$$\begin{aligned} g((1-\lambda)x + \lambda x') &= af((1-\lambda)x + \lambda x') + b \\ &\geqq a[(1-\lambda)f(x) + \lambda f(x')] + b \\ &= (1-\lambda)af(x) + \lambda af(x') + (1-\lambda)b + \lambda b \\ &= (1-\lambda)[af(x) + b] + \lambda[af(x') + b] \\ &= (1-\lambda)g(x) + \lambda g(x'). \end{aligned}$$

THEREFORE g IS CONCAVE.

SINCE $a > 0$, WE HAVE $af(x) = g(x) - b$, i.e., $f(x) = \frac{1}{a}g(x) - \frac{b}{a} = \alpha g(x) + \beta$ FOR $\alpha = \frac{1}{a}$, $\beta = -\frac{b}{a}$. THE ABOVE RESULT THEN SAYS THAT IF g IS CONCAVE, SO IS f. ||

(4) $f(x_1, x_2) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + x_1 x_2 - x_1 + x_2$.

(a) $f_1 = x_1 + x_2 - 1$, $f_2 = -x_2 + x_1 + 1$ $\nabla f = (x_1 + x_2 - 1, x_1 - x_2 + 1)$
$f_{11} = 1$, $f_{22} = -1$, $f_{12} = f_{21} = 1$ $D^2 f = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(b) $f_1 = 0$: $x_1 + x_2 - 1 = 0$ THE ONLY SOLUTION IS $x_1 = 0, x_2 = 1$.
$f_2 = 0$: $x_1 - x_2 + 1 = 0$ $\bar{x} = (0, 1)$ AND $f(\bar{x}) = \frac{1}{2}$.

(c) $P_2(\Delta x, \bar{x}) = f(\bar{x}) + \nabla f(\bar{x}) \Delta x + \frac{1}{2} \Delta x \, H(\bar{x}) \Delta x$

$= \frac{1}{2} + 0 + \frac{1}{2} \Delta x \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Delta x$, BECAUSE $\nabla f(\bar{x}) = (0,0)$

$= \frac{1}{2} + \frac{1}{2}\left[(\Delta x_1)^2 + 2 \Delta x_1 \Delta x_2 - (\Delta x_2)^2\right]$

$= \frac{1}{2} + \frac{1}{2}(\Delta x_1)^2 + \Delta x_1 \Delta x_2 - \frac{1}{2}(\Delta x_2)^2$.

(d) $\bar{x}$ IS A CRITICAL POINT OF f, SO WE NEED TO CHECK THE SECOND-ORDER CONDITIONS AT $\bar{x}$:

$|D^2 f(\bar{x})| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$, SO $D^2 f(\bar{x})$ IS INDEFINITE

AND THEREFORE $\bar{x}$ IS NEITHER A MAX OR MIN.

(e) $G(x_1, x_2) = x_1 + x_2 = 2$; $\nabla G = (1, 1)$.

FOC: $\nabla f = \lambda \nabla G$, i.e., $\quad 2 - 1 = \lambda$; $\lambda = 1$; $\therefore \bar{x}_1 = \bar{x}_2 = 1$.

$f_1 = \lambda G_1$: $x_1 + x_2 - 1 = \lambda$ $\quad x_1 + x_2 - 1 = x_1 - x_2 + 1$; i.e.,

$f_2 = \lambda G_2$: $x_1 - x_2 + 1 = \lambda$ $\quad x_2 - 1 = 1 - x_2$; $\therefore x_2 = 1, x_1 = \lambda$

SINCE $x_1 + x_2 = 1$, WE HAVE $x_1 = 1$.

$\nabla f(\bar{x}) = (1, 1)$, $\nabla G = (1, 1)$, $\lambda = 1$. SO $\bar{x} = (1, 1)$.

THE SECOND-ORDER CONDITIONS FOR $\bar{x}$ TO BE A LOCAL MAX OR MIN OF f s.t. $G(x)=2$:

SINCE $m=1$ AND $n=2$, WE NEED TO CHECK THE SIGN OF ONLY $|\mathbb{B}|$, WHERE $\mathbb{B}$ IS THE HESSIAN MATRIX OF f AT $\bar{x}$, BORDERED BY $\nabla G(\bar{x})$:

$$\mathbb{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}; \quad |\mathbb{B}| = 1+1-1+1=2,$$

SO THE QUADRATIC FORM $\Delta x H(x) \Delta x$ IS NEGATIVE DEFINITE AT $\bar{x}$, SUBJECT TO $x_1+x_2=2$, AND $\bar{x}$ IS THEREFORE A LOCAL MAXIMUM OF f SUBJECT TO $x_1+x_2=2$.

NOTE THAT THE BORDERED MATRIX $\mathbb{B}$ IS THE SAME AT EVERY x ON THE CONSTRAINT, SO THE QUADRATIC FORM $\Delta x H(x) \Delta x$, RESTRICTED TO Δx THAT KEEP x ON THE CONSTRAINT AND TO x ON THE CONSTRAINT, IS NEGATIVE DEFINITE. THEREFORE f, RESTRICTED TO THE CONSTRAINT, IS CONCAVE (ACTUALLY, STRICTLY CONCAVE), SO $\bar{x}$ IS A <u>GLOBAL</u> MAXIMUM SUBJECT TO THE CONSTRAINT.

(f) $G(x_1,x_2) = x_1 - x_2 = 2$: $\nabla G = (1,-1)$

FOC: $\nabla f = \lambda \nabla G$, i.e.,

$$\left.\begin{aligned} f_1 = \lambda G_1:&\quad x_1 + x_2 - 1 = \lambda \\ f_2 = \lambda G_2:&\quad x_1 - x_2 + 1 = -\lambda \end{aligned}\right\} \begin{aligned} &x_1 = 0, \therefore x_2 = -2, \lambda = -3. \\ &\bar{x} = (0,-2). \end{aligned}$$

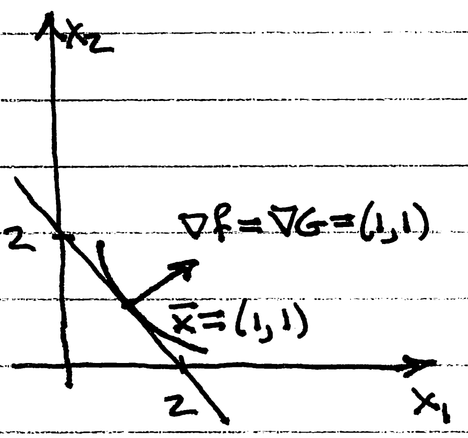
SECOND-ORDER CONDITIONS FOR $\bar{x}$ TO BE A LOCAL MAX OR MIN s.t. $G(x) = 2$:

WE AGAIN NEED TO CHECK THE SIGN OF ONLY $|\mathbb{B}|$:

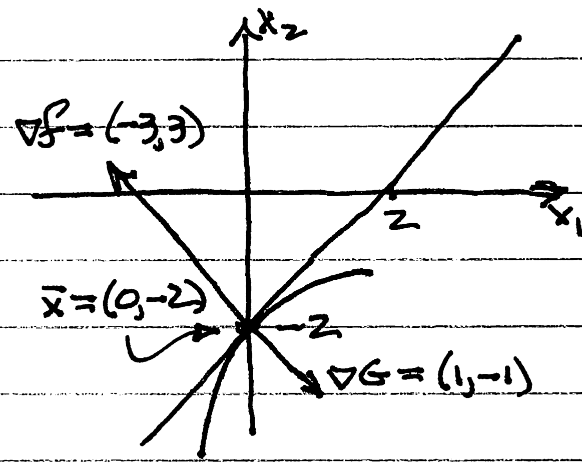
$$\mathbb{B} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}; \ |\mathbb{B}| = -1 - 1 - 1 + 1 = -2,$$

SO $\Delta x H(x) \Delta x$ IS POSITIVE DEFINITE SUBJECT TO THE ~~constraint~~ RESTRICTION THAT Δx LEAVES US ON THE CONSTRAINT $G(x) = 2$. THEREFORE $\bar{x}$ IS A LOCAL MINIMUM OF f SUBJECT TO $G(x) = 2$. AND JUST AS IN (e), $|\mathbb{B}|$ IS CONSTANT ON THE CONSTRAINT, SO $\Delta x H(x) \Delta x$ IS ~~NEGATIVE~~ POSITIVE DEFINITE SUBJECT TO $G(x) = 2$, AND THEREFORE $\bar{x}$ IS A GLOBAL MINIMUM SUBJECT TO $G(x) = 2$.

NOTE THAT (e) AND (f) CAN ALSO BE SOLVED BY CONVERTING THE FUNCTION f INTO A FUNCTION OF ONE VARIABLE: IN (e) WE HAVE $x_2 = 2 - x_1$, SO WE CAN DEFINE $g(x_1) = f(x_1, 2 - x_1)$, A QUADRATIC FUNCTION OF x_1, FOR WHICH IT'S EASY TO OBTAIN $g'(x_1)$ AND $g''(x_1)$. THE EXAM SAID, HOWEVER, THAT YOU WERE TO USE THE CONSTRAINED MAX/MIN CONDITIONS. BUT YOU COULD HAVE USED THIS METHOD TO CHECK YOUR SOLUTION.



(e)



(f)