ECON 519 FALL 2014
FINAL EXAM SOLUTIONS

(1) LET $x, y \in \bar{X}$, LET $0 < \alpha < 1$, AND LET $z = (1-\alpha)x + \alpha y$. WE WISH TO SHOW THAT $z \in \bar{X}$. BECAUSE $x, y \in \bar{X}$, THERE ARE SEQUENCES $\{x_n\}$ AND $\{y_n\}$ SUCH THAT $\forall n$: $x_n, y_n \in X$ AND $\lim x_n = x$ AND $\lim y_n = y$. FOR EACH n, LET $z_n = (1-\alpha)x_n + \alpha y_n$. THEN $z_n \in X$ FOR ALL n (BECAUSE X IS CONVEX), AND

$$\lim z_n = \lim [(1-\alpha)x_n + \alpha y_n] = (1-\alpha)\lim x_n + \alpha \lim y_n$$
$$= (1-\alpha)x + \alpha y = z.$$

SINCE $z_n \in X$ FOR ALL n AND $\lim z_n = z$, WE HAVE $z \in \bar{X}$.

(2) LET $x \in S^\circ$ — i.e., $x \in \bigcup\{V \subseteq S \mid V \text{ open}\}$. LET V BE ANY NONEMPTY OPEN SUBSET OF S SUCH THAT $x \in V$ — THERE MUST BE AT LEAST ONE SUCH SET, BECAUSE $x \in \bigcup\{V \subseteq S \mid V \text{ open}\}$. BY DEFINITION OF OPEN SETS, THERE IS AN OPEN BALL $B(x,r)$ ABOUT x SUCH THAT $B(x,r) \subseteq V \subseteq S$.

LET $B(x,r)$ BE AN OPEN BALL ABOUT x SUCH THAT $B(x,r) \subseteq S$. SINCE $B(x,r) \subseteq \bigcup\{V \subseteq S \mid V \text{ is open}\}$, WE HAVE $x \in S^\circ$.

(3) LET f BE CONTINUOUS (ACCORDING TO THE DEFINITION), LET V BE AN OPEN SET IN Y, AND LET $U = f^{-1}(V)$. WE MUST SHOW THAT U IS OPEN. LET $x \in U$ AND LET $y = f(x) \in V$. WE MUST SHOW THAT THERE IS AN $r > 0$ SUCH THAT $B(x,r) \subseteq U$. SUPPOSE NOT; THEN FOR EVERY $n \in \mathbb{N}$ THERE IS AN $x_n \in B(x, \frac{1}{n})$ FOR WHICH $x_n \notin U$, AND THEREFORE $f(x_n) \notin V$. SINCE V IS OPEN AND $f(x) \in V$, THE SEQUENCE $\{f(x_n)\}$ DOES NOT CONVERGE TO $f(x)$, CONTRADICTING THE CONTINUITY OF f.

FOR THE CONVERSE, ASSUME THAT FOR EVERY OPEN SET V IN Y, $f^{-1}(V)$ IS OPEN. LET $\{x_n\}$ BE A SEQUENCE THAT CONVERGES TO A POINT $x \in X$; LET $y = f(x)$ AND LET $y_n = f(x_n)$ FOR EACH n. WE MUST SHOW THAT $\lim y_n = y$. SUPPOSE $\{y_n\}$ DOES NOT CONVERGE TO y; THEN THERE IS AN OPEN SET $V \subseteq Y$ SUCH THAT $y \in V$ AND SUCH THAT A SUBSEQUENCE $\{y_{n_k}\}$ OF $\{y_n\}$ SATISFIES $\forall k: y_{n_k} \notin V$. LET $U = f^{-1}(V)$; THEN U IS AN OPEN SET, $x \in U$, AND $\forall k: x_{n_k} \notin U$. THEREFORE $\{x_n\}$ DOES NOT CONVERGE TO x, CONTRADICTING OUR ASSUMPTION THAT IT DOES.

(4) LET $S = (0,1) \subseteq \mathbb{R}$, LET $d(x,x') = |x - x'|$, AND LET $f(x) = \frac{1}{2}x$. CLEARLY f HAS NO FIXED POINT: $f(x) = \frac{1}{2}x = x$ ONLY AT $x = 0$, BUT $0 \notin S$. TO VERIFY THAT f IS A CONTRACTION, LET β SATISFY $\frac{1}{2} < \beta < 1$; THEN

$$\forall x, x' \in S: d(f(x), f(x')) = |\tfrac{1}{2}x - \tfrac{1}{2}x'| = \tfrac{1}{2}|x - x'| < \beta |x - x'|.$$

(5) (a) $x \sim x'$ IF AND ONLY IF $Ax = Ax'$.

$\sim$ IS REFLEXIVE: $Ax = Ax$, $\forall x \in \mathbb{R}^n$.

$\sim$ IS SYMMETRIC: $Ax = Ax' \Rightarrow Ax' = Ax$, $\forall x, x' \in \mathbb{R}^n$.

$\sim$ IS TRANSITIVE: $[Ax = Ax' \;\&\; Ax' = Ax''] \Rightarrow Ax = Ax''$, $\forall x', x'', x''' \in \mathbb{R}^n$.

(b) IF rank $A = n$, THEN A IS INVERTIBLE: FOR EVERY $b \in \mathbb{R}^n$ THE EQUATION $Ax = b$ HAS A UNIQUE SOLUTION — IF $Ax = b$ AND $Ax' = b$, THEN $x = x'$. THEREFORE $Ax = Ax' \Rightarrow x = x'$. IN OTHER WORDS, EACH EQUIVALENCE CLASS IS A SINGLETON, $[x] = \{x\}$.

(c) LET $n = 3$, AND ASSUME THAT rank $A = 1$. WLOG, LET THE FIRST ROW OF A BE $a \neq \underline{0}$ (BECAUSE rank $A > 0$); NOTE THAT $a \in \mathbb{R}^3$. BECAUSE rank $A = 1$, EACH OF THE REMAINING ROWS OF A IS A MULTIPLE OF a — i.e., $\exists \lambda_2, \lambda_3 \in \mathbb{R}$ SUCH THAT $\lambda_2 a$ AND $\lambda_3 a$ ARE THE SECOND AND THIRD ROWS OF A.

LET $x \in \mathbb{R}^3$; WE WANT TO DETERMINE $[x]$. LET $b = Ax$; IN PARTICULAR, $b_1 = a \cdot x$. AND ALSO $(\lambda_2 a) \cdot x = b_2 = \lambda_2 b_1$ AND $(\lambda_3 a) \cdot x = b_3 = \lambda_3 b_1$. LET $x' \in [x]$ — i.e. $Ax' = Ax = b$. IN PARTICULAR, $a \cdot x' = a \cdot x = b_1$ — i.e., $x' \in H(a, b_1)$, A HYPERPLANE CONTAINING x. THEREFORE WE HAVE $[x] \subseteq H(a, b_1)$.

NOW WE HAVE TO SHOW THAT $[x]$ IS THE WHOLE HYPERPLANE $H(a, b_1)$, NOT A PROPER SUBSET. LET $x' \in H(a, b_1)$ — i.e., $a \cdot x' = b_1$. THEN $(\lambda_2 a) \cdot x' = \lambda_2 b_1 = b_2$ AND $(\lambda_3 a) \cdot x' = \lambda_3 b_1 = b_3$ — i.e., $Ax' = b = Ax$, SO $x' \in [x]$, AND WE'VE ESTABLISHED THAT $H(a, b_1) \subseteq [x]$.

(6) THERE MUST BE EITHER AN INFINITE NUMBER OF TERMS OF $\{x_n\}$ (i.e., A SUBSEQUENCE) TO THE LEFT OF $\bar{x}$ ON THE REAL LINE, OR A SUBSEQUENCE ENTIRELY TO THE RIGHT OF $\bar{x}$ (OR BOTH), SO WLOG WE ASSUME THAT $\forall n \in \mathbb{N}: x_n < \bar{x}$. WE ALSO ASSUME WLOG THAT $\{x_n\}$ IS INCREASING, SINCE (WITH $x_n \to \bar{x}$ AND EACH $x_n < \bar{x}$) THERE MUST BE SUCH A SUBSEQUENCE.

LET N BE ANY $N \in \mathbb{N}$; FOR EVERY $n > N$ WE HAVE $x_N < x_n < \bar{x}$ — i.e., x_n IS A CONVEX COMBINATION $(1-\alpha_n)x_N + \alpha_n \bar{x}$ OF x_N AND $\bar{x}$ FOR SOME α_n. NOTE THAT $\lim \alpha_n = 1$, SINCE $\lim x_n = \bar{x}$. SINCE f IS CONCAVE, FOR EACH $n > N$ WE HAVE

$$f(x_n) = f((1-\alpha_n)x_N + \alpha_n \bar{x}) \geqq (1-\alpha_n) f(x_N) + \alpha_n f(\bar{x})$$

THEREFORE

$$\begin{aligned}
\bar{y} = \lim y_n = \lim f(x_n) &= \lim f((1-\alpha_n)x_N + \alpha_n \bar{x}) \\
&\geqq \lim \left[(1-\alpha_n) f(x_N) + \alpha_n f(\bar{x})\right] \\
&= \lim (1-\alpha_n) f(x_N) + \lim \alpha_n f(\bar{x}) \\
&= f(\bar{x}), \text{ BECAUSE } \lim \alpha_n = 1.
\end{aligned}$$