ECONOMICS 519, FALL 2011
FINAL EXAM SOLUTIONS

(1) LET $\{y_n\}$ BE A SEQUENCE IN $f(X)$; WE MUST SHOW THAT $\{y_n\}$ HAS A CONVERGENT SUBSEQUENCE. FOR EACH $n \in \mathbb{N}$, LET $x_n \in X$ BE s.t. $f(x_n) = y_n$; THEREFORE $\{x_n\}$ IS A SEQUENCE IN X. SINCE X IS COMPACT, $\{x_n\}$ HAS A CONVERGENT SUBSEQUENCE $\{x_{n_k}\}$ — LET $\bar{x} = \lim_{k\to\infty} x_{n_k}$. SINCE f IS CONTINUOUS, WE HAVE $\lim_{k\to\infty} f(x_{n_k}) = f(\bar{x})$ — i.e., $\{f(x_{n_k})\}$ IS A CONVERGENT SUBSEQUENCE OF $\{y_n\}$.

(2) (a) SUPPOSE $\{x_n\} \sim \{x_n'\}$ AND $\{x_n'\} \sim \{x_n''\}$; WE MUST SHOW THAT $\{x_n\} \sim \{x_n''\}$. WE HAVE

$$\forall \epsilon > 0 : \exists n_1 \in \mathbb{N} : m, k > n_1 \Rightarrow d(x_m, x_k') < \frac{\epsilon}{2}$$
$$\forall \epsilon > 0 : \exists n_2 \in \mathbb{N} : k, n > n_2 \Rightarrow d(x_k', x_n'') < \frac{\epsilon}{2}.$$

LET $\epsilon > 0$, AND LET $\bar{n} = \max\{n_1, n_2\}$; THEN FOR ANY $k > \bar{n}$ WE HAVE

$$d(x_m, x_n'') \leqq d(x_m, x_k') + d(x_k', x_n'') < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

THEREFORE $\{x_n\} \sim \{x_n''\}$.

(b) IF X IS COMPLETE, THEN EVERY CAUCHY SEQUENCE CONVERGES, AND THEREFORE EACH EQUIVALENCE CLASS IN $S/\sim$ CONSISTS OF ALL THE CAUCHY SEQUENCES IN X THAT CONVERGE TO THE SAME LIMIT. WE PROVE THIS AS FOLLOWS:
FIRST WE SHOW THAT IF $\{x_n'\} \sim \{x_n''\}$, THEN $\lim x_n' = \lim x_n''$. SUPPOSE NOT: $\lim x_n' = \bar{x} \neq \bar{\bar{x}} = \lim x_n''$.

LET $\epsilon = d(\bar{x}, \bar{\bar{x}})$ AND LET n^* BE SUCH THAT

$$m > n^* \Rightarrow d(x'_m, \bar{x}) < \frac{\epsilon}{3}$$

$$n > n^* \Rightarrow d(\bar{\bar{x}}, x''_n) < \frac{\epsilon}{3}$$

$$m, n > n^* \Rightarrow d(x'_m, x''_n) < \frac{\epsilon}{3}.$$

THEN
$$m, n > n^* \Rightarrow d(\bar{x}, \bar{\bar{x}}) \leq d(\bar{x}, x'_m) + d(x'_m, x''_n) + d(x''_n, \bar{\bar{x}})$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3};$$

i.e., $d(\bar{x}, \bar{\bar{x}}) < \epsilon$, WHICH CONTRADICTS THE FACT THAT $d(\bar{x}, \bar{\bar{x}}) = \epsilon$.

CONVERSELY, SUPPOSE $\{x_n\}, \{x'_n\} \in S$ AND $\lim x_n = \bar{x} = \lim x'_n$. WE WILL SHOW THAT $\{x_n\} \sim \{x'_n\}$. LET $\epsilon > 0$ AND LET $\bar{n}$ BE SUCH THAT

$$n > \bar{n} \Rightarrow \left[d(x_n, \bar{x}) < \frac{\epsilon}{2} \text{ AND } d(x'_n, \bar{x}) < \frac{\epsilon}{2} \right].$$

THEN

$$m, n > \bar{n} \Rightarrow d(x_m, \bar{x}) + d(x'_n, \bar{x}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

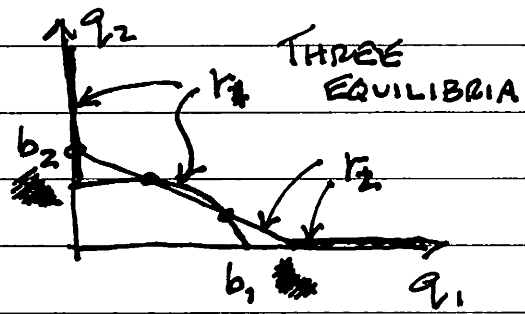
i.e., BY THE TRIANGLE INEQUALITY,

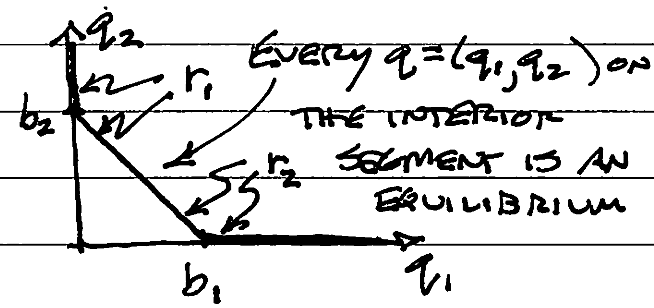
$$m, n > \bar{n} \Rightarrow d(x_m, x'_n) < \epsilon.$$

(3) (a) WRITE $B_1 = [0, b_1]$ AND $B_2 = [0, b_2]$, AND REDEFINE THE DOMAINS OF r_1 AND r_2: $r_1 : B_2 \to B_1$ AND $r_2 : B_1 \to B_2$. DEFINE $\bar{F} : B_1 \times B_2 \to B_1 \times B_2$ AS $\bar{F}(q_1, q_2) = (r_1(q_2), r_2(q_1))$. A COURNOT EQUILIBRIUM IS A $q^* = (q_1^*, q_2^*) \in B_1 \times B_2$ THAT SATISFIES $q_1^* = r_1(q_2^*)$ AND $q_2^* = r_2(q_1^*)$ — i.e., IT IS A FIXED POINT OF THE FUNCTION $\bar{F}$.

THE FUNCTION $\bar{F}$ IS CONTINUOUS (SINCE r_1 AND r_2 ARE CONTINUOUS). THE NONEMPTY SET $B_1 \times B_2$ IS COMPACT AND CONVEX (AS THE PRODUCT OF COMPACT, CONVEX SETS). THEREFORE BROUWER'S THEOREM APPLIES, ENSURING THAT $\bar{F}$ HAS A FIXED POINT — A COURNOT EQUILIBRIUM.

(b)





(c) THE BANACH (CONTRACTION MAPPING) THEOREM REQUIRES THAT THE DOMAIN BE A COMPLETE METRIC SPACE (WHICH $B_1 \times B_2$ IS) AND THAT $\bar{F}$ BE A CONTRACTION. THE GIVEN PROPERTIES OF r_1 AND r_2 AREN'T SUFFICIENT TO ENSURE THAT $\bar{F}$ IS A CONTRACTION, SO THIS THEOREM DOESN'T APPLY. INDEED, THE CONCLUSION OF THE THEOREM IS THAT THERE IS A <u>UNIQUE</u> FIXED POINT, BUT IN (b) WE HAVE FUNCTIONS r_1 AND r_2 THAT SATISFY THE ASSUMPTIONS THAT ARE GIVEN BUT FOR WHICH THERE ARE MULTIPLE EQUILIBRIA.

(4) ASSUME THAT $\{x_n\} \xrightarrow[a]{} \bar{x}$ AND ASSUME THAT $\|\cdot\|_a \sim \|\cdot\|_b$.
SINCE $\|\cdot\|_a \sim \|\cdot\|_b$, ~~Let~~ LET M BE SUCH THAT

$$\forall x \in V: \|x\|_b \leqq M\|x\|_a.$$

IN ORDER TO SHOW THAT $\{x_n\} \xrightarrow[b]{} \bar{x}$, LET $\epsilon > 0$; WE MUST SHOW THAT $\exists \bar{n}: n > \bar{n} \Rightarrow \|x_n - \bar{x}\|_b < \epsilon$. LET $r = \frac{1}{M}\epsilon$; SINCE $\{x_n\} \xrightarrow[a]{} \bar{x}$, WE KNOW THERE IS AN $\bar{n}$ s.t.

$$n > \bar{n} \Rightarrow \|x_n - \bar{x}\|_a < r,$$

AND THEREFORE $n > \bar{n} \Rightarrow \|x_n - \bar{x}\|_b < M\|x_n - \bar{x}\|_a < Mr = \epsilon$.

(5) LET $\{(z_n, p_n)\}$ BE A SEQUENCE THAT SATISFIES $p_n \in \mu(z_n)$, $\forall n$, AND WHICH CONVERGES TO $(\bar{z}, \bar{p})$.
WE MUST SHOW THAT $\bar{p} \in \mu(\bar{z})$.

(a) IF $\bar{z}_1 = \bar{z}_2$, THEN $\mu(\bar{z}) = S$, SO CLEARLY $\bar{p} \in \mu(\bar{z})$.

(b) IF $\bar{z}_1 < \bar{z}_2$, THEN $\exists \bar{n}: n > \bar{n} \Rightarrow z_{n1} < z_{n2}$, AND THEREFORE $n > \bar{n} \Rightarrow p_n = (0,1)$; THEREFORE $\bar{p} = (0,1) \in \mu(\bar{z})$.

(c) IF $\bar{z}_1 > \bar{z}_2$, THEN $\exists \bar{n}: n > \bar{n} \Rightarrow z_{n1} > z_{n2}$, AND THEREFORE $n > \bar{n} \Rightarrow p_n = (1,0)$; $\therefore \bar{p} = (1,0) \in \mu(\bar{z})$.

AN ALTERNATIVE PROOF USES THE MAXIMUM THEOREM:
S IS COMPACT; $p \cdot z : S \times \mathbb{R} \to \mathbb{R}$ IS CONTINUOUS;
$\varphi(z) = S$ FOR ALL $z \in \mathbb{R}$, SO φ IS CONTINUOUS.
THEREFORE μ HAS A CLOSED GRAPH, ACCORDING TO THE MAXIMUM THEOREM, BECAUSE

$$\mu(z) = \{p \in S \mid p \text{ MAXIMIZES } p \cdot z \text{ ON } S\}.$$