CAMBRIDGE SURVEYS OF ECONOMIC LITERATURE

APPLYING GENERAL EQUILIBRIUM

Applying general equilibrium

JOHN B. SHOVEN
Stanford University
and National Bureau of Economic Research

JOHN WHALLEY
University of Western Ontario
and National Bureau of Economic Research

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Introduction

The aim of this book is to make more widely available a body of recent research activity that has become known as applied general equilibrium analysis. The central idea underlying this work is to convert the Walrasian general equilibrium structure (formalized in the 1950s by Kenneth Arrow, Gerard Debreu, and others) from an abstract representation of an economy into realistic models of actual economies. Numerical, empirically based general equilibrium models can then be used to evaluate concrete policy options by specifying production and demand parameters and incorporating data reflective of real economies.

We have earlier summarized a number of these modeling efforts in a survey article (Shoven and Whalley 1984). Here we try to go one stage further and give readers more of a sense of how to do their own modeling, including developing an appropriate equilibrium structure, calibrating their model, compiling counterfactual equilibria, and interpreting results. The first part of the book develops the techniques required to apply general equilibrium theory to policy evaluations. The second part presents a number of applications we have made in our previous research.

The Walrasian general equilibrium model provides an ideal framework for appraising the effects of policy changes on resource allocation and for assessing who gains and loses, policy impacts that are not well covered by empirical macro models. In this volume, we outline a number of ways in which applied versions of this model are providing fresh insights into long-standing policy controversies.

Our use of the term "general equilibrium" corresponds to the well-known Arrow–Debreu model, elaborated in Arrow and Hahn (1971). The number of consumers in the model is specified. Each consumer has an initial endowment of the N commodities and a set of preferences, resulting in demand functions for each commodity. Market demands are the sum of each consumer's demands. Commodity market demands depend on all prices, and are continuous, nonnegative, homogeneous of degree zero (i.e., no money illusion), and satisfy Walras's law (i.e., that at any set of prices, the total value of consumer expenditures equals consumer incomes. On the production side, technology is described by either constant-returns-to-scale activities or nonincreasing-returns-to-scale production functions. Producers maximize profits. The zero homogeneity of demand functions and the linear homogeneity of profits in prices (i.e., doubling all prices doubles money profits) imply that only relative prices are of any significance in such a model. The absolute price level has no impact on the equilibrium outcome.

Equilibrium in this model is characterized by a set of prices and levels of production in each industry such that the market demand equals supply for all commodities (including disposals if any commodity is a free good). Since producers are assumed to maximize profits, this implies that in the constant-returns-to-scale case, no activity (or cost-minimizing technique for production functions) does any better than break even at the equilibrium prices.

Most contemporary applied general equilibrium models are numerical analogs of traditional two-sector general equilibrium models popularized by James Meade, Harry Johnson, Arnold Harberger, and others in the 1950s and 1960s. Earlier analytic work with these models has examined the distortionary effects of taxes, tariffs, and other policies, along with functional incidence questions. More recent applied models, including those discussed here, provide numerical estimates of efficiency and distributional effects within the same framework.

The value of these computational general equilibrium models is that numerical simulation removes the need to work in small dimensions, and much more detail and complexity can be incorporated than in simple analytic models. For instance, tax-policy models can simultaneously accommodate several taxes. This is important even when evaluating changes in only one tax because taxes compound in effect with other taxes. Also, use of a tax-policy model permits an evaluation of comprehensive tax-reform proposals such as those debated in the United States during the 1984–6 period. Likewise, the complexities of the issues handled in trade negotiations in the General Agreement on Tariffs and Trade (GATT), such as simultaneous tariff reductions in several countries or codes to limit the use of nontariff barriers, cannot be analyzed in ways useful to policy makers other than through numerical techniques. Models involving 30 or more sectors and industries are commonly employed, providing substantial detail for policy makers concerned with feedback effects of policy initiatives directed at specific products or industries.

In the next chapter we briefly review the theory of general equilibrium relevant for applied general equilibrium analysis. We sketch proofs of existence, and discuss in detail the inclusion of such policy instruments as taxes and tariffs for which a modeling of government behavior is also required. The applied models that follow in later chapters are consistent with the Arrow–Debreu theoretical structure, reflecting the attempt in applied general equilibrium work to make that structure relevant to policy.

The techniques and models described in this book have been applied to a range of policy questions in a number of economic fields over the last ten or so years. These include public finance and taxation issues, international trade-policy questions, evaluations of alternative development strategies, the implications of energy policies, regional questions, and even issues in macroeconomic policy.

Policy makers daily confront the need to make decisions on all manner of both major and minor policy matters that affect such issues as the intersectoral allocation of resources and the distribution of income. Some form of numerical model is implicit in the actions of any policy maker. Techniques such as those presented here can, in our opinion, help policy makers by making explicit the implications of alternative courses of action within a framework broadly consistent with that currently accepted by many microeconomic theorists. Although model results are not precise owing to data and other problems, they nonetheless provide a vehicle for generating initial null hypotheses on the impacts of policy changes where none previously existed. They also yield assessments of the impacts of policies, which may challenge the received wisdom that guides policy making. We emphasize the large elements of subjective judgment involved both in building and in using these models, and also their large potential for generating fresh insights on policy issues of the day.

We hope that the insights gained from particular models will become clearer as the reader proceeds with the description of the various models, but some examples may be helpful at this stage. One result of applied general equilibrium tax models' use has been a reassessment of the importance of the efficiency costs of taxes relative to their equity consequences. Twenty years ago it was commonly believed that the resource misallocation costs of taxes were relatively small (perhaps 1% of GNP), and that the tax system in total did little to redistribute income. The applied models have challenged this view by producing estimates of combined welfare costs from distortions in the tax system of 8%–10% of GNP, and estimates of their marginal welfare costs as large as $0.50 per additional dollar of revenue raised. These models have also indicated that there are more significant redistribution effects caused by the tax system than had previously been believed. The models have also been used to provide a ranking of various tax-policy alternatives, and have showed how interactions among the various parts of the tax system can affect the evaluation of tax-reform alternatives.

Applied general equilibrium trade models that assume constant returns to scale have in the main suggested that the welfare costs of trade distortions are smaller than those of tax distortions, confirming the suggestions made by previous partial equilibrium calculations. Such trade models have, however, found a significantly different geographical pattern in results owing to terms-of-trade effects. When increasing returns to scale are incorporated along with market structure features, larger effects are discovered.

Further insights have been gained in cases involving more specific analyses. For instance, in analyzing the impacts of regional trade agreements, results suggest that the more important effects arise from the elimination of trade barriers in partner countries and the benefits from improved access abroad, rather than from the internal trade creation and trade diversion effects discussed in the theoretical literature. Analyses of the impact of protection in the North on developing countries have put the annual costs to the South at about the value of the aid flow from the North, suggesting that these aid and trade effects roughly cancel out. (See Section 8.4.)

Another example of a model-generated insight concerns the international trade dimensions of the basis used for indirect taxes by American trading partners. Given that these taxes are heavier on manufactures than nonmanufactures, model results have shown that a destination basis abroad (taxes on imports, but not on exports) may be better for the United States than an origin basis (taxes on exports, but not on imports). This follows if the United States is a net importer of manufactures in its trade with the country involved. This suggests that an origin basis abroad need not be in the U.S. interest, as is often assumed; nor should the United States push for the same basis in all its trading partners.

These and other insights could, no doubt, have been obtained in other ways, but the virtue of using applied general equilibrium models is that, once constructed, they yield a facile tool for analyzing a wide range of possible policy changes. Such analyses generate results that either yield an initial null hypothesis, or challenge the prevailing view. It may be that subsequently the conclusions from the model are rejected as inappropriate; the assumptions may be considered unrealistic, errors may be unearthed, or other factors may undermine confidence in the results. But there will be situations in which the modeler and those involved in the policy decision process will have gained new perspectives as a result of using the model. In our opinion, this is the virtue of the approach, and is the reason why we believe its use in the policy process will spread further than the applications we report.

Applied general equilibrium analysis is not without its own problems. As the development of applied general equilibrium models has progressed from merely demonstrating the feasibility of model construction and solution to serious policy applications, a variety of issues has arisen. Most modelers recognize the difficulties of parameter specification and the necessity for (possibly contentious) assumptions. Elasticity and other key parameter values play a pivotal role in all model outcomes, and no consensus exists regarding numerical values for most of the important elasticities. The choice of elasticity values is frequently based on scant empirical evidence, and what evidence exists is often contradictory. This limits the degree of confidence with which model results can be held. On the other hand, there are no clearly superior alternative models available to policy makers who base their decisions on efficiency and distributional consequences of alternative policy changes. Whether partial equilibrium, general equilibrium, or back-of-the-envelope quantification is used, key parameter values must be selected, yet current econometric literature in so many of the areas involved is not particularly helpful.

Modelers have also been forced to confront the problem of model preselection: the need to specify key assumptions underlying the particular applied model to be used before any model calculations can begin. Both theoretical and applied modelers have long recognized the need to use particular assumptions in building general equilibrium models, assumptions such as full employment and perfect competition. There are also other equally important assumptions that enter these analyses. One example involves international factor flows. In tax models, the incidence effects of capital income taxes are substantially affected by the choice of this assumption: If capital is internationally mobile, capital owners will not bear the burden of income taxes; in a closed economy, however, domestic capital owners may well be affected. Another example is the treatment of time. In a static model, a tax on consumption may appear distorting since capital goods are tax free, but this effect will be absent when the tax is analyzed from an intertemporal viewpoint.

A further difficulty with general equilibrium analysis is how the policies themselves are represented in applied models. Taxes must be represented in model-equivalent form, and yet for each tax there is substantial disagreement in the literature as to the appropriate treatment. In the case of the corporate tax, for instance, the original treatment adopted by Harberger (1962) of assuming average and marginal tax rates on capital income by industry to be the same can bias results. Recent literature has emphasized that this tax could be viewed as applying to only the equity return on capital rather than to the total return, that is, as a tax on one financing instrument available to firms. This view has been used by Stiglitz (1973) to argue that the tax is a lump-sum tax; more recently, Gordon (1981) has argued that the corporate tax is in effect a benefit-related risk-sharing tax. Similar difficulties arise in other areas of application. With trade models, for instance, the modeling of nontariff barriers is an especially difficult and contentious issue.

A final and somewhat broader issue is that most of the applied general equilibrium models are not tested in any meaningful statistical sense. Parameter specification usually proceeds using deterministic calibration (often to one year's data), and there is no statistical test of the model specification (see Mansur and Whalley 1984). In determining parameter values by calibrating to a single data observation, equilibrium features in the data are emphasized. A purely deterministic equilibrium model in which consumers maximize utility and producers maximize profits is thus constructed in a manner consistent with the observed economy. With enough flexibility in choosing the form of the deterministic model, one can always choose a model so as to fit the data exactly. Econometricians, who are more accustomed to thinking in terms of models whose economic structure is simple but whose statistical structure is complex (rather than vice versa), frequently find this a source of discomfort.

PART I

Techniques

2

General equilibrium theory

2.1 Introduction

Applying general equilibrium analysis to policy issues requires a basic understanding of general equilibrium theory, which we attempt to provide in this chapter. A general equilibrium model of an economy can be best understood as one in which there are markets for each of N commodities, and consistent optimization occurs as part of equilibrium. Consumers maximize utility subject to their budget constraint, leading to the demand-side specification of the model. Producers maximize profits, leading to the production-side specification. In equilibrium, market prices are such that the required equilibrium conditions hold. Demand equals supply for all commodities, and in the constant-returns-to-scale case zero-profit conditions are satisfied for each industry.

A number of basic elements can be identified in general equilibrium models. In a pure exchange economy, consumers have endowments and demand functions (usually derived from utility maximization). In the two-consumer–two-good case, this leads to the well-known Edgeworth box analysis of general equilibrium of exchange. In the case of an economy with production, endowments and demands are once again specified, but production sets also need to be incorporated into the analysis.

2.2 Structure of general equilibrium models

The simple pure trade general equilibrium model can be represented as one in which there are N commodities, $1, \ldots, N$, each of which has a nonnegative price $p_i \geq 0$. Market prices are denoted by the vector $\mathbf{p} = p_1, \ldots, p_N$. The term W_i represents the nonnegative economywide endowment of commodity i owned by consumers, assumed to be strictly positive for at least one i; $\xi_i(\mathbf{p})$ are the market demand functions, which

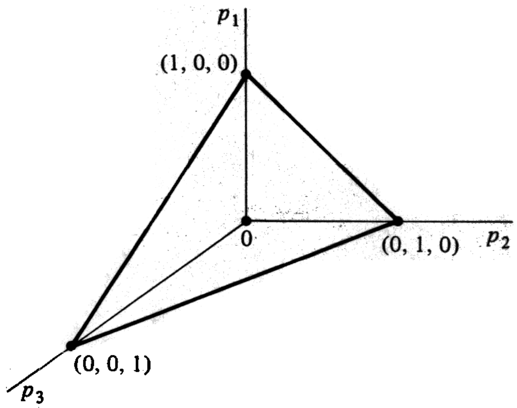

Figure 2.1. A 3-dimensional unit price simplex.

are nonnegative, continuous, and homogeneous of degree zero in $\mathbf{p}$. The latter assumption implies that doubling all prices doubles incomes and hence the physical quantities demanded are unchanged.

Because the demand functions are assumed to be homogeneous of degree zero in prices, an arbitrary normalization of prices can be used; we will ordinarily set

$$\sum_{i=1}^{N} p_i = 1. \tag{2.1}$$

The prices of the N commodities lie on a unit simplex. The case where $N=3$ is depicted in Figure 2.1.

A key further assumption usually made on the market demands is that they satisfy Walras's law. Walras's law states that the value of market demands equals the value of the economy's endowments, that is,

$$\sum_{i=1}^{N} p_i \xi_i(\mathbf{p}) = \sum_{i=1}^{N} p_i W_i, \tag{2.2}$$

or the value of market excess demands equals zero at all prices,

$$\sum_{i=1}^{N} p_i(\xi_i(\mathbf{p}) - W_i) = 0. \tag{2.3}$$

This condition must hold for any set of prices, whether or not they are equilibrium prices. Walras's law is an important basic check on any equilibrium system; if it does not hold, a misspecification is usually present

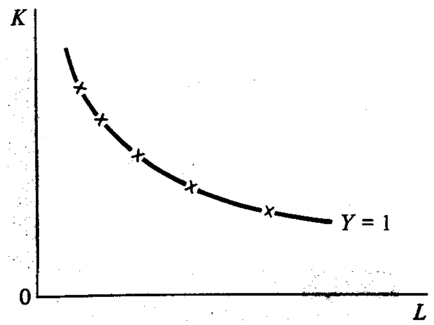

Figure 2.2. Approximating a unit isoquant by a series of linear activities.

since the model of the economy in question violates the sum of individual budget constraints.

A general equilibrium in this system is a set of prices p_i^* such that

$$\xi_i(\mathbf{p}^*) - W_i \leq 0, \tag{2.4}$$

with equality if $p_i^* > 0$. Equilibrium prices, therefore, clear markets.

A general equilibrium model with production is similar, but would also include a specification of a production technology. One representation of production has a finite number K of constant-returns-to-scale activities or methods of production. Each activity is described by coefficients a_{ij} denoting the use of good i in activity j when the activity is operated at unit intensity. A negative sign indicates an input and a positive sign an output.

These activities can be displayed in the nonsquare matrix A, which lists the many possible ways of producing commodities and can be used in any nonnegative linear combination:

$$A = \begin{bmatrix} -1 & 0 & & 0 & a_{1,N+1} & \cdots & a_{1,j} & \cdots & a_{1,K} \\ 0 & -1 & & 0 & \cdot & & \cdot & & \cdot \\ \cdot & 0 & \cdots & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & 0 & \cdot & & \cdot & & \cdot \\ 0 & \cdot & & -1 & a_{N,N+1} & \cdots & a_{N,j} & \cdots & a_{N,K} \end{bmatrix}$$

The first N activities are "slack" activities reflecting the possibility of free disposal of each commodity. In the case of only two inputs (capital and labor) and one output, these technology activities can be thought of as approximating a unit isoquant through a series of activities giving linear facet isoquants, as shown in Figure 2.2.

Activities are assumed to be nonreversible; that is, it is not possible to produce inputs from outputs. The vector $\mathbf{X} = X_1, \ldots, X_K$ denotes levels of intensity of operation associated with each activity and is nonnegative. Production is assumed to be bounded; that is, infinite amounts of outputs from finite inputs are ruled out. This corresponds to the often-made "no free lunch" assumption. In technical terms, this assumption implies that the set of $\mathbf{X}$ such that

$$\sum_{j=1}^{K} a_{ij} X_j + W_i \geq 0 \quad \text{for all } i \tag{2.6}$$

is contained within a bounded set.

A general equilibrium for this model is given by a set of prices p_i^* and activity levels X_j^* such that:

(i) demands equal supplies,

$$\xi_i(\mathbf{p}^*) = \sum_{j=1}^{K} a_{ij} X_j^* + W_i \quad \text{for all } i = 1, \ldots, N; \tag{2.7}$$

and

(ii) no production activity makes positive profits, whereas those in use break even,

$$\sum_{i=1}^{N} p_i^* a_{ij} \leq 0 \quad (=0 \text{ if } X_j^* > 0) \quad \text{for all } j = 1, \ldots, K. \tag{2.8}$$

In contrast to the pure exchange equilibrium model, no complementary slackness condition appears in equilibrium condition (i) because of the incorporation of the disposal activities in the technology matrix A. If there is excess supply of any commodity, disposal occurs through the use of a disposal activity.

2.3 Existence of a general equilibrium

The major result of postwar mathematical general equilibrium theory has been to demonstrate the existence of such an equilibrium by showing the applicability of mathematical fixed point theorems to economic models. This is the essential contribution of Arrow and Debreu (1954), which has been expanded upon in Debreu (1959), Arrow and Hahn (1971), and elsewhere. Since applying general equilibrium models to policy issues involves computing equilibria, these fixed point theorems are important: It is essential to know that an equilibrium exists for a given model before attempting to compute that equilibrium.

Fixed point theorems involve continuous mappings of the unit simplex into itself. That is, if S denotes the set of vectors $\mathbf{X}$ on the unit simplex

$$\sum_{i=1}^{N} X_i = 1, \quad X_i \geq 0, \tag{2.9}$$

then the mapping $F(\mathbf{X})$ is such that

$$\sum_{i=1}^{N} F_i(\mathbf{X}) = 1, \quad F_i(\mathbf{X}) \geq 0, \tag{2.10}$$

and F satisfies continuity properties. Two different types of mappings are usually considered: point-to-point mappings (i.e., $F(\mathbf{X})$ is a point on the unit simplex), and point-to-set mappings (i.e., $F(\mathbf{X})$ is a set on the unit simplex). These are displayed in Figure 2.3, where fixed points under each type of mappings are also presented.

The two basic fixed point theorems used in general equilibrium theory are the Brouwer fixed point theorem for point-to-point mappings, and the Kakutani fixed point theorem for point-to-set mappings. These are discussed by Scarf (1973, p. 28), who states them as follows:

Brouwer's theorem: Let $Y = F(\mathbf{X})$ be a continuous function mapping the simplex into itself; then there exists a fixed point of the mapping, that is, a vector such that $\mathbf{X}^* = F(\mathbf{X}^*)$.

Kakutani's theorem: Let the point-to-set mapping $\mathbf{X} \to \phi(\mathbf{X})$ of the simplex S into itself be upper semicontinuous. Assume that for each $\mathbf{X}$, $\phi(\mathbf{X})$ is a nonempty, closed, convex set. Then there exists a fixed point $\hat{\mathbf{X}} \in \phi(\hat{\mathbf{X}})$.

The concept of upper semicontinuity in Kakutani's fixed point theorem is as follows: A point-to-set mapping $\mathbf{X} \to \phi(\mathbf{X})$ is upper semicontinuous if the following condition is satisfied. Let $\mathbf{X}^1, \mathbf{X}^2, \ldots, \mathbf{X}^j$ converge to $\mathbf{X}$. Let $\theta^1 \in \phi(\mathbf{X}^1), \theta^2 \in \phi(\mathbf{X}^2), \ldots$ and assume that the sequence $\theta^1, \theta^2, \ldots, \theta^j, \ldots$ converges to θ. Then $\theta \in \phi(\mathbf{X})$. The importance of the continuity property can be depicted diagrammatically by considering a mapping of the unit interval into itself, as in Figure 2.4. Provided the mapping is continuous, a fixed point must exist. However, if a discontinuity occurs in the mapping, a fixed point need not exist. In this simple case, a continuous mapping must cross the 45° line for a fixed point to exist. The existence of more than one fixed point is illustrated in Figure 2.5.

An example of a continuous mapping of the unit simplex into itself would transform each point x on the unit simplex into the midpoint of the line segment connecting x to the center of the simplex; that is,

$$y_i = \frac{x_i + 1/n}{2} \quad \text{for all } i. \tag{2.11}$$

The fixed point in this case is clearly the center of the simplex. A second example is to let A be an $N \times N$ nonnegative matrix whose column and

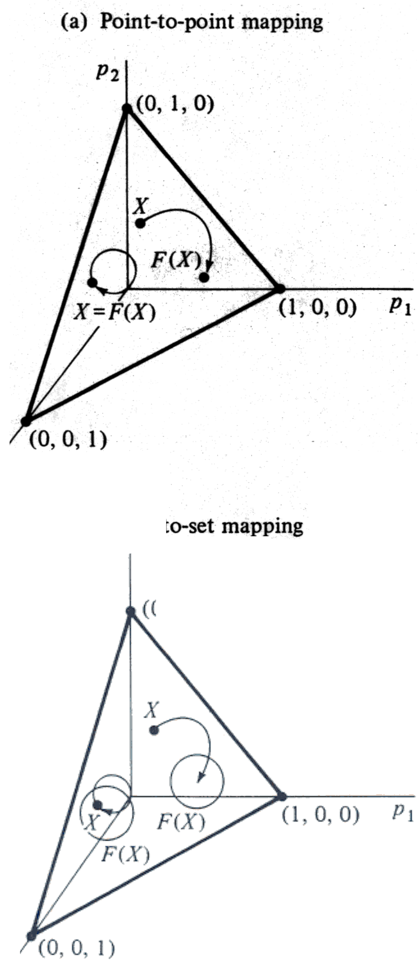

Figure 2.3. Point-to-point and point-to-set mappings of the unit sim-
plex into itself.

ow sums equal unity. The equation $y = Ax$ maps the simplex into itself. both of these cases, appealing to the Brouwer fixed point theorem es-blishes the existence of a fixed point.

rther example, important to general equilibrium models, is one and Nikaido (1956) that transforms the excess deman

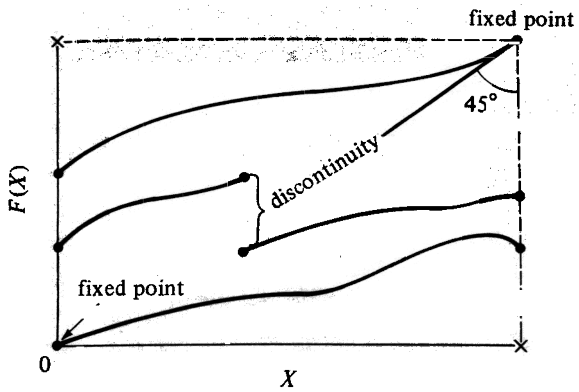

Figure 2.4. A mapping of unit interval into itself.

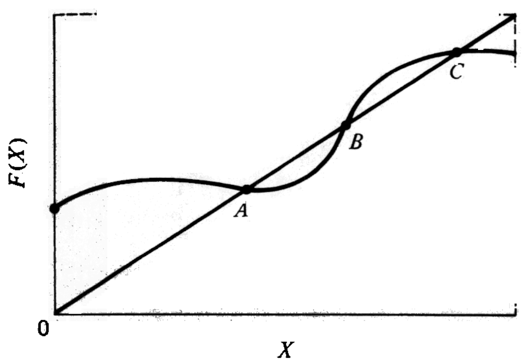

Figure 2.5. Existence of multiple fixed points.

functions of the pure exchange model into a mapping of the unit simplex into itself, to which the Brouwer fixed point theorem may be applied. A fixed point of this mapping can then be shown to imply that the equilibrium conditions of the pure exchange model must hold.

This mapping works as follows: An equilibrium for the pure exchange model is a vector of prices $\mathbf{p}^*$ such that $\xi_i(\mathbf{p}^*) - W_i \leq 0$ for all i, with strict equality holding if $p_i^* > 0$. Defining the excess demand functions $g_i(\mathbf{p}) = \xi_i(\mathbf{p}) - W_i$, the Gale–Nikaido mapping is

$$y_i = \frac{p_i + \max[0, g_i(\mathbf{p})]}{1 + \sum_{i=1}^{N} \max[0, g_i(\mathbf{p})]} \quad \text{for all } i = 1, \ldots, N. \tag{2.12}$$

Because $\sum_{i=1}^{N} y_i = 1$, $y_i \geq 0$, and $g_i(\mathbf{p})$ is continuous (since the demand functions $\xi_i(\mathbf{p})$ are assumed continuous), (2.12) provides a continuous mapping of the unit simplex into itself. By Brouwer's theorem there must exist a fixed point, that is, a vector $\mathbf{p}^*$ such that

$$p_i^* = \frac{p_i^* + \max[0, g_i(\mathbf{p}^*)]}{1 + \sum_{i=1}^{N} \max[0, g_i(\mathbf{p}^*)]} \quad \text{for all } i = 1, \ldots, N. \tag{2.13}$$

To show that $\mathbf{p}^*$ corresponds to an equilibrium for this model, we define $c = 1 + \sum_{i=1}^{N} \max[0, g_i(\mathbf{p}^*)]$. Equation (2.13) can then be rewritten as

$$cp_i^* = p_i^* + \max[0, g_i(\mathbf{p}^*)] \quad \text{for all } i. \tag{2.14}$$

Thus, if $c > 1$,

$$(c-1)p_i^* = \max[0, g_i(\mathbf{p}^*)] \quad \text{for all } i. \tag{2.15}$$

If $p_i^* > 0$, this implies $g_i(\mathbf{p}^*) > 0$. However, by Walras's law we know that

$$\sum_{i=1}^{N} p_i^* g_i(\mathbf{p}^*) = 0. \tag{2.16}$$

Since $p_i^* > 0$ implies $g_i(\mathbf{p}^*) > 0$ for the case where $c > 1$, it must be true that $\sum_{i=1}^{N} p_i^* g_i(\mathbf{p}^*) > 0$; that is, Walras's law is violated. Thus, $c > 1$ implies a contradiction. By definition $c \geq 1$, and hence c must equal 1. From (2.13) this implies that $g_i(\mathbf{p}^*) \leq 0$ for all i, and this together with Walras's law means that all of the conditions for equilibrium hold at the fixed point $\mathbf{p}^*$.

The demonstration of the existence of equilibria is more complicated when production activities are included in the model. If these exhibit decreasing returns to scale and the technology is such that there is a unique profit-maximizing output level for each commodity for any vector of prices, then the Gale–Nikaido mapping just presented can again be used. However, if constant returns to scale are assumed, or if several production plans are equally profitable, then there is not a unique vector of production in response to any set of prices.

This situation is most easily handled by applying the Kakutani fixed point theorem for point-to-set mappings as discussed in Debreu (1959). Here we present an outline of the full general equilibrium existence proof. Many of the intermediate assertions are not rigorously proven, since they can be found in Debreu (1959) for the cases without taxes and in Shoven (1974) for the cases with taxes.

In a general equilibrium model with production, we assume that the role of producers is to choose and carry out a production plan. Such a plan is characterized by the specification of all of the producer's inputs, represented by negative numbers, and outputs, given as positive numbers. Hence, a production plan can be represented as a point in N-dimensional Euclidean space.

The economy is assumed to consist of L producers, each of which (say, the lth) has an associated feasible production set Y^l. The assumptions made with regard to Y^l for each l are as follows.

(A.2.1) Y^l is convex (i.e., if $\mathbf{y}^1 \in Y^l$ and $\mathbf{y}^2 \in Y^l$, then $\alpha\mathbf{y}^1 + (1-\alpha)\mathbf{y}^2 \in Y^l$ for $0 \le \alpha \le 1$).

(A.2.2) $0 \in Y^l$ (i.e., inactivity is feasible).

(A.2.3) Y^l is closed (i.e., if $\mathbf{y}^n \in Y^l$ and $\mathbf{y}^n \to \mathbf{y}^0$, then $\mathbf{y}^0 \in Y^l$).

(A.2.4) Y^l is bounded from above (i.e., for some B, $\mathbf{y} \le B$ for all $\mathbf{y} \in Y^l$).

Assumptions (A.2.1) and (A.2.2) imply that the production possibility set Y is convex and contains the origin. Existence proofs can be produced with somewhat more general assumptions, particularly with respect to (A.2.4), but the general methodology still follows that presented here.

We assume that each producer l maximizes profit subject to the production set Y^l. Thus, for a price vector $\mathbf{p} = (p_1, \ldots, p_N)$, the producer finds those activities $\mathbf{y}^* \in Y^l$ such that

$$\sum_{i=1}^{N} p_i y_i^* \ge \sum_{i=1}^{N} p_i y_i \quad \text{for all } \mathbf{y} \in Y^l. \tag{2.17}$$

Let $Y^l(\mathbf{p})$ be the point-to-set supply response of the producer, defined as

$$Y^l(\mathbf{p}) = \left\{ \mathbf{y}^* \,\middle|\, \mathbf{y}^* \in Y^l,\ \sum_{i=1}^{N} p_i y_i^* \ge \sum_{i=1}^{N} p_i y_i \text{ for all } \mathbf{y} \in Y^l \right\}. \tag{2.18}$$

Thus, $Y^l(\mathbf{p})$ is the set of all profit-maximizing responses for producer l to the price vector $\mathbf{p}$. Debreu (1959) shows that (A.2.1)–(A.2.4) imply that the set $Y^l(\mathbf{p})$ is nonnull, convex, and closed, and further that the mapping $Y^l(\mathbf{p})$ is upper semicontinuous. The market production response $Y(\mathbf{p})$ is defined as the sum of the individual production responses

$$Y(\mathbf{p}) = \sum_{l=1}^{L} Y^l(\mathbf{p}), \tag{2.19}$$

and retains all of the properties of the individual production response sets $Y^l(\mathbf{p})$.

If we define π_l as the profit of the lth producer,

$$\pi_l(\mathbf{p}) \equiv \max_{\mathbf{y} \in Y^l} \sum_{i=1}^{N} p_i y_i, \tag{2.20}$$

it can further be demonstrated that, under (A.2.1)–(A.2.4), $\pi_l(\mathbf{p})$ is a continuous, nonnegative function of prices $\mathbf{p}$, as are total profits in the economy $\pi(\mathbf{p}) = \sum_{l=1}^{L} \pi_l(\mathbf{p})$.

The consumer side of this model is the same as in the pure exchange model just presented. However, in this case formal assumptions analogous to our treatment of production are required. We assume that there are M consumers, who may be thought of as individuals, households, or even classes of individuals. The consumption of each consumer is represented by an N-dimensional vector, and the set of all possible consumption vectors for the mth consumer is denoted by X^m.

We let x_i^m denote the consumption of commodity i by consumer m. Leisure is treated as a consumed good rather than as a factor of production, so that X^m is a nonnegative set (i.e., $x_i^m \geq 0$, $i = 1, \ldots, N$, $m = 1, \ldots, M$). In addition, $0 \in X^m$, so that the consumer chooses from a set bounded from below by the origin. We also specify a utility function $U^m(\mathbf{x}^m)$ for each individual.

The assumptions for the consumer side of the model are as follows.

(A.2.5) X^m is convex and closed.

(A.2.6) $U^m(\mathbf{x}^m)$ is continuous and semistrictly quasiconcave (i.e., the "at least as desired as" set is convex). If $\mathbf{x}^1$ is preferred to $\mathbf{x}^2$, then $[\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2]$, $0 < \alpha \leq 1$, is also preferred to $\mathbf{x}^2$.

(A.2.7) Individual m's income is given by the value of m's initial endowments, plus m's share of production profits.

That is,

$$I^m(\mathbf{p}) = \sum_{i=1}^{N} p_i w_i^m + \mu^m(\mathbf{p}), \tag{2.21}$$

where w_i^m is individual m's initial endowment of commodity i and $\mu^m(\mathbf{p})$ is the amount of profits distributed to m. The function $\mu^m(\mathbf{p})$ is assumed to be nonnegative, continuous, and homogeneous of degree one in $\mathbf{p}$, so $\mu^m(\mathbf{p})$ represents nominal profits and doubles should all prices double. All profits are disbursed to consumers; that is,

$$\sum_{m=1}^{M} \mu^m(\mathbf{p}) = \pi(\mathbf{p}) = \sum_{l=1}^{L} \pi_l(\mathbf{p}). \tag{2.22}$$

The sum of the initial endowments of individual consumers equals the economy's endowment. That is,

$$\sum_{m=1}^{M} w_i^m = W_i \quad \text{for } i = 1, \ldots, N. \tag{2.23}$$

We also make the assumptions

(A.2.8) $w^m > \xi^m$ for some $\xi^m \in X^m$

and

(A.2.9) X^m is bounded[1] (i.e., no individual can consume more than some given finite amount of each commodity).

[1] The boundedness assumption (A.2.9) can also be replaced by a nonsatiation assumption.

Using (A.2.5)–(A.2.9), we can define consumer m's demand correspondence as

$$X^m(\mathbf{p}) \equiv \{\mathbf{x}^m \mid \mathbf{x}^m \text{ maximizes } U^m(\mathbf{x}^m) \\ \text{subject to } C^m(\mathbf{x}^m, \mathbf{p}) \leq I^m(\mathbf{p}),\ \mathbf{x}^m \in X^m\}, \tag{2.24}$$

where

$$C^m(\mathbf{x}^m, \mathbf{p}) = \sum_{i=1}^{N} p_i x_i^m \tag{2.25}$$

is the cost of the consumption vector $\mathbf{x}^m$ at prices $\mathbf{p}$. It can be shown that for any $\mathbf{p}$, $X^m(\mathbf{p})$ is a nonnull, closed, convex, and bounded set, and that the mapping $X^m(\mathbf{p})$ is upper semicontinuous. The market demand response $X(\mathbf{p})$, defined as the sum of the individual demand responses, is given by

$$X(\mathbf{p}) = \sum_{m=1}^{M} X^m(\mathbf{p}), \tag{2.26}$$

which also has the same properties as the individual responses.

To prove that there exists an equilibrium price vector for this model, we first define the market excess demand mapping (or correspondence) as

$$Z(\mathbf{p}) = X(\mathbf{p}) - Y(\mathbf{p}) - W. \tag{2.27}$$

The vector $\mathbf{p}^*$ then defines an equilibrium set of prices if there is a $z \in Z(\mathbf{p}^*)$ such that

$$z_i \leq 0 \quad \text{for all } i = 1, \ldots, N. \tag{2.28}$$

That is, there are demand and supply responses consistent with $\mathbf{p}^*$ for which all markets clear. Walras's law implies that $z_i < 0$ only when $p_i^* = 0$.

In order to prove the existence of an equilibrium price vector $\mathbf{p}^*$, the first step is to establish that $Z(\mathbf{p})$ is nonnull, closed, convex, and bounded for any $\mathbf{p}$, and that $Z(\mathbf{p})$ is an upper semicontinuous correspondence. These properties are all implied by the corresponding properties of $X(\mathbf{p})$ and $Y(\mathbf{p})$.

Now let Z be a closed, convex, and bounded set that contains all of the sets $Z(\mathbf{p})$, where $\mathbf{p}$ is on the unit simplex. In order to use Kakutani's theorem to prove the existence of an equilibrium, we must use a higher-dimensional product space than that represented by the unit simplex. If we let S be the set of points lying on the unit simplex, we consider the product set $S \times Z$. This set is closed, convex, and bounded, and meets the required conditions for Kakutani's theorem. What we need to do is describe an upper semicontinuous point-to-set mapping of this set into itself, such that a fixed point is an economic equilibrium.

Consider the mapping of each point $(\mathbf{p}, \mathbf{z})$ in the product space $S \times Z$ that operates in the following manner:

$$\begin{pmatrix} \mathbf{p} & \mathbf{p}(\mathbf{z}) \\ \mathbf{z} & Z(\mathbf{p}) \end{pmatrix}. \tag{2.29}$$

The first N elements of $(\mathbf{p}, \mathbf{z})$ are the N prices, the corresponding set for which (the last N elements of the image set) is the market excess demand correspondence $Z(\mathbf{p})$ just described. The vector $\mathbf{z}$, which contains the last N elements of the vector $(\mathbf{p}, \mathbf{z})$, are market excess demands.

The image set for $\mathbf{z}$ is the set of price vectors $\mathbf{p}(\mathbf{z})$ that maximize the value of the market excess demands. That is, they form the solution set to the problem

$$\max \sum_{i=1}^{N} p_i z_i, \quad \text{where} \quad \sum_{i=1}^{N} p_i = 1, \;\; p_i \geq 0. \tag{2.30}$$

The product mapping $(\mathbf{p}, \mathbf{z}) \rightarrow (\mathbf{p}(\mathbf{z}), Z(\mathbf{p}))$ is an upper semicontinuous mapping of $S \times Z$ into itself, and thus by Kakutani's theorem there is a fixed point. That is, there is a vector $(\mathbf{p}^*, \mathbf{z}^*)$, where $\mathbf{p}^* \in \mathbf{p}(\mathbf{z}^*)$ and $\mathbf{z}^* \in Z(\mathbf{p}^*)$.

The remaining task in demonstrating the existence of equilibrium is to show that $\mathbf{p}^*$ represents an equilibrium price vector. To do so, we recall that Walras's law can be written

$$\sum_{i=1}^{N} p_i^* z_i^* = 0. \tag{2.31}$$

However, since $\mathbf{p}^* \in \mathbf{p}(\mathbf{z}^*)$, we know that

$$0 = \sum_{i=1}^{N} p_i^* z_i^* \geq \sum_{i=1}^{N} p_i z_i^* \quad \text{for } \mathbf{p} \in S, \tag{2.32}$$

where S is the unit simplex. This inequality must hold for all $\mathbf{p} \in S$, including the cases where $\mathbf{p}$ has all its coordinates equal to zero except for the kth, which is equal to unity. For this $\mathbf{p}$, we have

$$z_k^* \leq 0. \tag{2.33}$$

But since this argument is valid for all k between 1 and N, we see that all excess demands must be nonpositive, the very equilibrium condition we were seeking. This completes the sketch of the proof of the existence of an equilibrium in a model with production.

The application of fixed point theorems to show the existence of an equilibrium for pure exchange models and those with production were important in the 1950s and 1960s in demonstrating the consistency of general equilibrium models. They provide the logical support for the subsequent use of this framework for policy analysis.

The weakness of such applications is twofold. First, they provide nonconstructive rather than constructive proofs of the existence of equilibrium; that is, they show that equilibria exist but do not provide techniques by which equilibria can actually be determined. Second, existence per se has no policy significance. Policy makers are interested in how the economy will behave when policy or other variables change; their interest is in comparative statics rather than existence. Thus, fixed point theorems are only relevant in testing the logical consistency of models prior to the models' use in comparative static policy analysis; such theorems do not provide insights as to how economic behavior will actually change when policies change. They can only be employed in this way if they can be made constructive (i.e., be used to find actual equilibria). The extension of the Brouwer and Kakutani fixed point theorems in this direction is what underlies the work of Scarf (1967, 1973) on fixed point algorithms, which we describe in the next chapter.

2.4 Extending existence proofs to models with taxes and tariffs

Section 2.3 outlined existence proofs for the conventional Arrow–Debreu model in which no government policy interventions occur. Because of the central role of policy analysis in recent applications of general equilibrium techniques, we now turn to adding taxes, tariffs, and trade restrictions to this model in order to make it more appropriate for the kind of governmental-policy evaluations we are seeking. As taxes and tariffs are important institutional realities of real-world economies, their inclusion in a general equilibrium framework is of some importance. The existence property is reassuring in the sense that without it one cannot seriously contemplate developing general algorithms for the computation of such equilibria that can in turn be useful for economic analyses.

We will first discuss taxes. The inclusion of taxes necessitates the introduction of a government into the traditional general equilibrium model. The role of the government is to be interpreted as solely that of a tax-collecting and revenue-dispersing agency. It may, however, disperse some of the revenue to itself (i.e., retain it) in order to buy goods and services, but the issues involving the derivation of a government utility function, which determines government purchases, are not dealt with here. For simplicity, we discuss the case where all government revenue is distributed to consumers as transfer payments. Each consumer's disbursement is a function of total government revenue, and this is what makes the inclusion of taxes into the general equilibrium model a nontrivial extension of the basic Arrow–Debreu model. The individual agents are now interdependent since incomes are partially determined by government revenue, which in turn is a function of every agent's decisions. Prices no longer convey enough information for an individual consumer to determine his or her demands, as consumer income is dependent not only on prices but also on the demands of all other consumers and the production of all producers. It is this feature – not the additional notational complexity caused by having different agents face different effective (i.e., after-tax) prices – that makes the problem interesting. The solution involves working in a space of slightly higher dimensionality than is customary, with more information than simply prices being communicated between agents.

Rather than repeat the logical sequence of the proof sketched in Section 2.3, we will merely indicate how taxes modify the basic existence argument. Since we have already characterized the problem posed by taxes as one where prices no longer convey sufficient information, the nature of the solution – namely, to augment the N-dimensional price vector $\mathbf{p}$ – should not be surprising. The added variable in the case where there is one government (the national government) is total government revenue R, generating an $(N+1)$-dimensional simplex containing vectors $\bar{\mathbf{p}}$, $\bar{\mathbf{p}} = (p_1, \ldots, p_N, R)$.

Since taxes drive a wedge between the prices that buyers pay and what sellers receive, it is important to clarify what the announced prices signify. We take them to be the prices faced by consumers before consumer taxes. They can be thought of as "consumer selling prices": the prices consumers receive for their initial endowments when they sell to producers or other consumers. Thus, these are the prices producers pay for inputs before any taxes that may apply, and the prices at which they sell their outputs (including any producer-output taxes). If a producer faces taxes on both inputs and outputs, that producer will pay more than the corresponding p_i if good i is an input and receive (net) less than the price p_i if good i is an output. Likewise, if consumers face consumption taxes, they must pay more than p_i for the ith good. As before, we are searching for an equilibrium vector, although in this case it is the augmented price vector $\bar{\mathbf{p}}^*$. As both the demand and supply correspondences to be defined are homogeneous of degree zero in $\bar{\mathbf{p}}$, the search for $\bar{\mathbf{p}}^*$ will be confined to

$$\bar{S} \equiv \left\{ \bar{\mathbf{p}} \,\middle|\, \sum_{i=1}^{N} p_i = 1,\ 0 \le R \le \bar{R} \right\}, \tag{2.34}$$

where $\bar{R}$ is a large number the magnitude of which is discussed later. Alternatively, the search could take place on the surface of an $(N+1)$-dimensional unit simplex with the units of revenue chosen in such a way as to make it somewhat comparable in magnitude to the commodity prices.

Each of the L producers has associated with it a given set of *ad valorem* tax rates $t^l = (t_1^l, \ldots, t_N^l)$ on its production activity. The assumptions regarding the set of feasible production activities Y^l are the same as for the no-tax case, that is, assumptions (A.2.1)–(A.2.4) with one additional assumption as follows:

(A.2.10) For any i where $t_i^l > 0$, if $y_i^l < 0$ and $\mathbf{y}^l \in Y^l$ then there is no $\mathbf{y} \in Y^l$ with $y_i > 0$.

This assumption simply rules out the case where there are two activities in the production set Y^l, one of which produces a taxed commodity as an output while the other uses the same commodity as an input, and eliminates the possibility of avoiding taxation through vertical integration. Producers may be defined more narrowly here (i.e., for tax purposes) than in other contexts.

It is assumed that each producer l maximizes profit subject to the production set Y^l. Thus, for a given augmented price vector $\bar{\mathbf{p}} = (p_1, \ldots, p_N, R)$ a producer finds those activities $\mathbf{y}^* \in Y^l$ such that

$$\sum_{i=1}^{N} p_i(y_i^* - t_i^l |y_i^*|) \geq \sum_{i=1}^{N} p_i(y_i - t_i^l |y_i|) \quad \text{for all } \mathbf{y} \in Y^l. \tag{2.35}$$

Let $Y^l(\bar{\mathbf{p}})$ be the supply response of the firm, defined as

$$Y^l(\bar{\mathbf{p}}) \equiv \left\{ \mathbf{y}^* \,\middle|\, \mathbf{y}^* \in Y^l, \sum_{i=1}^{N} p_i(y_i^* - t_i^l |y_i^*|) \geq \sum_{i=1}^{N} p_i(y_i - t_i^l |y_i|) \text{ for all } \mathbf{y} \in Y^l \right\}. \tag{2.36}$$

In Shoven (1974) it is shown that $Y^l(\bar{\mathbf{p}})$ is nonnull, convex, closed, and bounded for any fixed $\bar{\mathbf{p}}$, and also that the mapping $Y(\bar{\mathbf{p}})$ is upper semicontinuous. The proofs follow closely those in Debreu (1959). An $(N+1)$-dimensional set $\bar{Y}^l(\bar{\mathbf{p}})$, termed the *tax-augmented production response* of the lth producer, can be defined as

$$\bar{Y}^l(\bar{\mathbf{p}}) = \left\{ (\mathbf{y}, \tau) \,\middle|\, \mathbf{y} \in Y^l(\bar{\mathbf{p}}),\ \tau = -\sum_{i=1}^{N} p_i t_i^l |y_i| \right\}. \tag{2.37}$$

That is, each $\mathbf{y} \in Y^l(\bar{\mathbf{p}})$ is augmented by the negative of the amount of tax the lth producer pays given $\mathbf{y}$ and $\bar{\mathbf{p}}$. With assumptions (A.2.1)–(A.2.4) and (A.2.10), $\bar{Y}^l(\bar{\mathbf{p}})$ is also nonnull, convex, closed, and bounded for each $\bar{\mathbf{p}}$, and $\bar{Y}^l(\bar{\mathbf{p}})$ is a bounded upper semicontinuous mapping.

The *market-augmented production correspondence* is defined as the sum of the individual augmented production responses. That is,

$$\bar{Y}(\bar{\mathbf{p}}) = \sum_{l=1}^{L} \bar{Y}^l(\bar{\mathbf{p}}). \tag{2.38}$$

Under (A.2.1)–(A.2.4) and (A.2.10), $\bar{Y}(\bar{\mathbf{p}})$ has all of the properties of $\bar{Y}^l(\bar{\mathbf{p}})$.

The profit of the lth producer is given by

$$\pi_l(\bar{\mathbf{p}}) \equiv \max_{\mathbf{y} \in Y^l} \sum_{i=1}^{N} p_i(y_i - t_i^l |y_i|). \tag{2.39}$$

As in the case without taxes, $\pi_l(\bar{\mathbf{p}})$ is a continuous and nonnegative function, as is the function giving total profits in the economy $\pi(\bar{\mathbf{p}}) = \sum_{l=1}^{L} \pi_l(\bar{\mathbf{p}})$.

The consumer side of the model also needs relatively slight modification for the inclusion of taxes. As with producers, we allow each of the consumers to face a different set of commodity tax rates, $\mathbf{s}^m = (s_1^m, \ldots, s_N^m)$. The assumptions dealing with consumption (A.2.5)–(A.2.9) are left unchanged except that individual m's income is now given by

$$I^m(\bar{\mathbf{p}}) = \sum_{i=1}^{N} p_i w_i^m + \mu^m(\bar{\mathbf{p}}) + r^m(\bar{\mathbf{p}}), \tag{2.40}$$

where the new third term is the amount of government revenue distributed to m as transfer payments. The $r^m(\bar{\mathbf{p}})$ function is assumed to be nonnegative, continuous, and homogeneous of degree one. We also assume that

$$\sum_{m=1}^{M} r^m(\bar{\mathbf{p}}) = R. \tag{2.41}$$

The consumer's demand correspondence $X^m(\bar{\mathbf{p}})$ is as without taxes (i.e., m finds the sets of feasible consumption bundles that maximize utility), except that the cost of $\mathbf{x}^m$ for consumer m at prices $\mathbf{p}$ is given by

$$C^m(\mathbf{x}^m, \mathbf{p}) = \sum_{i=1}^{N} p_i x_i^m + S^m(\mathbf{x}^m, \mathbf{p}), \tag{2.42}$$

where

$$S^m(\mathbf{x}^m, \mathbf{p}) = \sum_{i=1}^{N} p_i s_i^m x_i^m \tag{2.43}$$

is the total purchase taxes paid by consumer m. The correspondence $X^m(\bar{\mathbf{p}})$ retains all of the properties of the no-tax $X^m(\mathbf{p})$.

The tax-augmented demand response is an $(N+1)$-dimensional set $\bar{X}^m(\bar{\mathbf{p}})$ defined as

$$\bar{X}^m(\bar{\mathbf{p}}) \equiv \{(\mathbf{x}, \gamma) \mid \mathbf{x} \in X^m(\bar{\mathbf{p}}),\ \gamma = S^m(\mathbf{x}, \mathbf{p})\}. \tag{2.44}$$

The market-augmented demand response is the sum of the individual $\bar{X}^m$ sets, and both the market-augmented response $\bar{X}(\bar{\mathbf{p}})$ and the individual augmented responses $\bar{X}^m(\bar{\mathbf{p}})$ are nonnull, closed, convex, and bounded, and are upper semicontinuous mappings.

Equilibrium in this case has the properties that

$$\sum_{m=1}^{M} \mathbf{x}^{m*} \leq \sum_{l=1}^{L} \mathbf{y}^{l*} + W \tag{2.45}$$

and

$$\sum_{m=1}^{M} S^m(\mathbf{x}^{m^*}, \mathbf{p}^*) + \sum_{l=1}^{L} \sum_{i=1}^{N} p_i^* t_i^l |y_i^{l^*}| = R^* \tag{2.46}$$

for some $\mathbf{x}^{m^*} \in X^m(\bar{\mathbf{p}}^*)$ for all m, and for some $\mathbf{y}^{l^*} \in Y(\bar{\mathbf{p}})$ for all l. That is, in addition to excess demands being nonnegative, in equilibrium revenue collections match the announced government revenue R^*.

With this structure, the proof of existence is straightforward. Let $\bar{W}$ be the augmented vector of the economy's endowments,

$$\bar{W} \equiv (W_1, \ldots, W_N, 0), \tag{2.47}$$

and let $\bar{Z}(\bar{\mathbf{p}})$ be the market-augmented excess demand correspondence

$$\bar{Z}(\bar{\mathbf{p}}) \equiv \bar{X}(\bar{\mathbf{p}}) - \bar{Y}(\bar{\mathbf{p}}) - \bar{W}; \tag{2.48}$$

$\bar{Z}$ retains the nonnull, closed, convex, and bounded properties of $\bar{X}(\bar{\mathbf{p}})$ and $\bar{Y}(\bar{\mathbf{p}})$, and is an upper semicontinuous mapping.

To prove the existence of an equilibrium in an economy such as this, one gain resorts to a product space. Let $\bar{Z}$ be a closed, convex, and bounded set that contains all of the sets $\bar{Z}(\bar{\mathbf{p}})$, $\bar{\mathbf{p}} \in \bar{S}$. Then we show that a fixed point of a mapping of the product space $\bar{S} \times \bar{Z}$ into itself will define an economic equilibrium. The map can be pictured as

$$\begin{pmatrix} \bar{\mathbf{p}} \\ \bar{\mathbf{z}} \end{pmatrix} \times \begin{pmatrix} \bar{\mathbf{p}}(\bar{\mathbf{z}}) \\ \bar{Z}(\bar{\mathbf{p}}) \end{pmatrix}, \tag{2.49}$$

where we have just developed the $\bar{Z}(\bar{\mathbf{p}})$ correspondence. The first N components of the $\bar{\mathbf{p}}(\bar{\mathbf{z}})$ correspondence represent the set of price vectors that maximize the value of market excess demands, as in the no-tax case. The last component of $\bar{\mathbf{p}}(\bar{\mathbf{z}})$ is equal to $\bar{\mathbf{z}}_{N+1}$. That is, the last dimension of the $\bar{\mathbf{p}}(\bar{\mathbf{z}})$ correspondence is simply the identity function.

This mapping of the $\bar{S} \times \bar{Z}$ space into itself meets all of the conditions of Kakutani's theorem, and thus there exists some $(\bar{\mathbf{p}}^*, \bar{\mathbf{z}}^*)$ such that

$$\bar{\mathbf{p}}^* \in \bar{\mathbf{p}}(\bar{\mathbf{z}}^*) \quad \text{and} \quad \bar{\mathbf{z}}^* \in \bar{Z}(\bar{\mathbf{p}}^*). \tag{2.50}$$

The fact that $\mathbf{p}^* \in \bar{\mathbf{p}}(\bar{\mathbf{z}}^*)$ implies that, for the last dimension, we have

$$p_{N+1}^* = R^* = z_{N+1}^* = \sum_{m=1}^{M} S^m(\mathbf{x}^{m^*}, \mathbf{p}^*) + \sum_{l=1}^{L} \sum_{i=1}^{N} p_i^* t_i^l |y_i|; \tag{2.51}$$

that is, the second condition for an equilibrium holds. The argument that all market excess demands are nonpositive is completely analogous to the no-tax case of Section 2.3. Therefore, there exists an equilibrium for an economy with both consumers and producers facing arbitrary and differentiated tax vectors.

The above arguments can easily be extended to cover a model in which there are a number, say K, of governments. This model can be interpreted either as an international trade model where several countries trade with each other, or as a model with tiered governments (e.g., federal/state/local). In fact, the two interpretations can be merged to yield a model with several countries each of which has governments within governments. Again, government is considered to be simply a tax-collecting and revenue-distributing agent.

With several governments, say K, the system of prices must be augmented by K revenue terms and may be represented by $\bar{\mathbf{p}} = (p_1, \ldots, p_N, R_1, \ldots, R_K)$. Similar goods in different locations are treated as separate commodities. The definition of location can be as narrow as necessary. That is, if any agent is taxed by any government at different tax rates on two physically identical commodities because of their location, then these two items are treated as different commodities. As before, it is assumed that there are a total of N commodities, M consumers, and L producers in the model (i.e., the "world").

Each of the M consumers is assigned a claim on one or more of the K revenue terms in such a way that

$$\sum_{m=1}^{M} r_k^m(\bar{\mathbf{p}}) = R_k \quad \text{for } k = 1, \ldots, K, \tag{2.52}$$

where $r_k^m(\bar{\mathbf{p}})$ is a continuous, linear, homogeneous function representing individual m's distribution from government k. A special case would be where each individual has claim to the revenue of only one government; in this case, the mth consumer's income is given by

$$I^m(\bar{\mathbf{p}}) = \sum_{i=1}^{N} p_i w_i^m + \mu^m(\bar{\mathbf{p}}) + \sum_{k=1}^{K} r_k^m(\bar{\mathbf{p}}). \tag{2.53}$$

Each consumer faces a set of tax rates $\mathbf{s}^m = (s_1^m, \ldots, s_N^m)$, as before. It makes no difference to the consumer whether part or all of these taxes are termed tariffs. Further, the consumer is indifferent as to which government or combination of governments is imposing these taxes. The mechanics of the existence proof allow complete generality in that each of the K governments could tax individual m on purchases of each of the N goods (of course, realism may dictate that many of these tax rates are zero). The individual tax vectors imposed on individual m sum to a total tax vector; that is,

$$s_i^m = \sum_{k=1}^{K} {}^k s_i^m \quad \text{for } m = 1, \ldots, M, \; i = 1, \ldots, N, \tag{2.54}$$

where ${}^k s_i^m$ is the tax rate imposed by the kth government on the mth consumer's purchases of the ith commodity.

Let ${}^k s^m(\mathbf{x}^m, \mathbf{p})$, the taxes the mth consumer pays to the kth government for consumption $\mathbf{x}^m$ at prices $\mathbf{p}$, be defined as

$$ {}^k s^m(\mathbf{x}^m, \mathbf{p}) = \sum_{i=1}^{N} p_i {}^k s_i^m p_i^* s_i^m x_i^m. \tag{2.55}$$

The set $\bar{X}^m(\bar{\mathbf{p}})$, consumer m's augmented demand response, is then redefined as the $(N+K)$-dimensional set

$$\bar{X}^m(\bar{\mathbf{p}}) \equiv \{(\mathbf{x}, \gamma_1, \ldots, \gamma_K) \mid \mathbf{x} \in X^m(\bar{\mathbf{p}}),\ \gamma_k = {}^k s^m(\mathbf{x}, \mathbf{p});\ k = 1, \ldots, K\}. \tag{2.56}$$

With the same assumptions as before, $\bar{X}^m(\bar{\mathbf{p}})$ is nonnull, closed, convex, and bounded for any $\bar{\mathbf{p}}$, and $\bar{X}^m(\bar{\mathbf{p}})$ is an upper semicontinuous mapping. The market-augmented demand response is defined as

$$\bar{X}(\bar{\mathbf{p}}) \equiv \sum_{m=1}^{M} \bar{X}^m(\bar{\mathbf{p}}). \tag{2.57}$$

The analysis on the production side is quite symmetrical to that on the consumer side. Each producer, say the lth, faces a vector of tax rates $\mathbf{t} = (t_1^l, \ldots, t_N^l)$, which is the sum of the vectors of tax rates imposed on producer activity by the K governments. That is,

$$t_i^l = \sum_{k=1}^{K} {}^k t_i^l \quad \text{for } l = 1, \ldots, L,\ i = 1, \ldots, N, \tag{2.58}$$

where ${}^k t_i^l$ is the tax rate imposed by the kth government on the use of the ith commodity by the lth producer. Naturally, there are special cases where many of the tax rates are zero.

The set $\bar{Y}^l(\bar{\mathbf{p}})$, producer l's augmented production correspondence, is redefined as the $(N+K)$-dimensional set

$$\bar{Y}^l(\bar{\mathbf{p}}) \equiv \left\{(\mathbf{y}, \tau_1, \ldots, \tau_K) \,\middle|\, \mathbf{y} \in Y^l(\bar{\mathbf{p}}),\ \tau_k = -\sum_{i=1}^{N} p_i {}^k t_i^l |y_i|;\ k = 1, \ldots, K\right\}. \tag{2.59}$$

As redefined, $\bar{Y}^l(\bar{\mathbf{p}})$ retains all of its properties of the one-government case. The market-augmented production correspondence is again defined as

$$\bar{Y}(\bar{\mathbf{p}}) \equiv \sum_{l=1}^{L} \bar{Y}^l(\bar{\mathbf{p}}). \tag{2.60}$$

Letting $\bar{W}$ be an $(N+K)$-dimensional vector

$$\bar{W} = (w_1, \ldots, w_N, 0, \ldots, 0), \tag{2.61}$$

$\bar{Z}(\bar{\mathbf{p}})$ is then defined as

$$\bar{Z}(\bar{\mathbf{p}}) \equiv \bar{X}(\bar{\mathbf{p}}) - \bar{Y}(\bar{\mathbf{p}}) - \bar{W}. \tag{2.62}$$

An equilibrium in this model is a $\bar{\mathbf{p}}^*$ such that there exists an $\mathbf{x}^{m^*} \in X^m(\bar{\mathbf{p}}^*)$ and a $\mathbf{y}^* \in Y(\bar{\mathbf{p}}^*)$ with the properties

$$\sum_{m=1}^{M} \mathbf{x}^{m^*} \leq \sum_{l=1}^{L} \mathbf{y}^{l^*} + W \tag{2.63}$$

and

$$\sum_{m=1}^{M} {}^{k}S^m(\mathbf{x}^{m^*}, \mathbf{p}^*) + \sum_{l=1}^{L} \sum_{i=1}^{N} p_i^* \, {}^{k}t_i^l |y_i^{l^*}| = R_k^*, \quad k = 1, \ldots, K. \tag{2.64}$$

That is, at $\mathbf{x}^{m^*}$ $(m=1,\ldots,M)$; $\mathbf{y}^{l^*}$ $(l=1,\ldots,L)$; and $\bar{\mathbf{p}}^*$, demand is less than supply for all N commodities and tax collections equal revenue distributed for each of the K governments.

The proof of existence for the multigovernment case is very similar to the one-government case. Both $\bar{S}$ and $\bar{Z}$ are now $(N+K)$-dimensional, and the last K dimensions of the $(N+K)$-dimensional correspondence $\bar{\mathbf{p}}(\bar{\mathbf{z}})$ have K identity functions. The result is an equilibrium where all excess demands are nonpositive and all governments have balanced budgets.

Trade policy, price-control schemes, additional taxes, or other policy interventions can be incorporated into the general equilibrium model and existence demonstrated. Wealth taxes and profit taxes are added in Shoven (1974). It would also be simple to incorporate Social Security and payroll taxes. Quotas and other quantity restrictions can also be added. A particularly easy way to do this is to create an artificial commodity, termed "tickets," which must be purchased when consuming or using the quantity-constrained commodity, as discussed in Shoven and Whalley (1972). The fact that these tickets must be allocated as endowments and may have a positive value simply reflects the fact that real rents are created if a government agency creates an artificial scarcity. Price-control schemes are closely related to tax schemes, as noted by Imam and Whalley (1982). In this case, controlling consumer prices while allowing producer prices to be endogenously determined means that the wedge between them is not prespecified as in the tax case.

2.5 Two-sector general equilibrium models

A more specific form of general equilibrium model, widely used in the applied fields of taxation and international trade, is the two-sector general equilibrium model (see the discussion in Ch. 8 of Atkinson and Stiglitz 1980, and Jones 1965). This model is used where the focus is on comparative static analysis for policy evaluation, rather than on existence. Since much of the theoretical literature in applied policy fields uses two-sector models, it is natural for numerical modelers to use a similar structure, so results can be checked against theoretical work. Also, much of the data available for use with applied models (such as national accounts data, input–output data, and other sources) fits a two-sector modeling approach.

Considerable simplification of equilibrium solutions can also be gained in applications by exploiting the special structure of two-sector models, since factors are not produced and commodities are not initially owned. It is possible to compute an equilibrium solution for these models by searching only in the space of factor prices and using zero-profit conditions to determine goods prices. In effect, an equilibrium in these models can be found by working with derived factor excess demands alone. Existence can also be shown using these functions. Even though two-sector models involve production, these simplifications allow Brouwer's theorem to be used in the proof of existence. Such dimension-reducing techniques are also important for the discussion of computational methods in the next chapter.

Following Uzawa (1963), the two-sector model, based on an assumption of constant returns to scale in each industry, can be defined by a five-equation system.

Production functions in each sector:

$$Y_i = F_i(K_i, L_i) = L_i f_i(k_i) \quad (i=1,2). \tag{2.65}$$

Value of marginal factor products equalized across sectors:

$$\frac{P_i}{P_j} = P = \frac{f_j'}{f_i'} = \frac{f_j - k_j f_j'}{f_i - k_i f_i'} \quad (i=1,2,\ j \neq i). \tag{2.66}$$

Wage–rentals ratio equals the ratio of marginal factor products in each sector:

$$\omega = \frac{f_i}{f_i'} - k_i \quad (i=1,2). \tag{2.67}$$

Cost-minimizing factor input ratio in each sector as a function of the wage–rentals ratio:

$$\frac{\partial k_i}{\partial \omega} = \frac{-(f_i')^2}{f_i f_i''} > 0 \quad (i=1,2). \tag{2.68}$$

Cost-covering commodity price ratio as a function of the wage–rentals ratio:

$$\frac{1}{P} \cdot \frac{\partial P}{\partial \omega} = \left(\frac{1}{k_i + \omega} - \frac{1}{k_j + \omega} \right) \quad (i=1,2,\ j \neq i). \tag{2.69}$$

Equations (2.65) define constant-returns-to-scale, two-input production functions in each industry. Using these, the marginal products of factors can be written in terms of the intensive-form production function $(f(k)=F(K/L,1))$ as

$$\frac{\partial F_i}{\partial K_i}=f_i'(k_i),\ \frac{\partial F_i}{\partial L_i}=f_i(k)-k_i f_i' \quad (i=1,2). \tag{2.70}$$

Equation (2.66) thus gives the relative cost-covering commodity prices consistent with competitive goods and factor markets, and (2.67) is the equilibrium condition that factor prices in each of the two sectors equal the ratio of marginal factor products. Equation (2.68) describes the behavior of the cost-minimizing capital–labor ratio for each sector with respect to the wage–rentals ratio, and (2.69) describes how competitive output prices change as factor prices change. Demand conditions in the algebraic statement in this model are usually not explicitly specified, although when used for numerical computation (as will be seen later) the precise demand conditions assumed determine the particular equilibrium attained.

Of these five equations, the first three require no explanation, but (2.68) and (2.69) involve some manipulation to derive them. The derivations proceed as follows.

Derivation of equation (2.68): From (2.67),

$$\omega=\frac{f_i-k_i f_i'}{f_i'} \quad (i=1,2); \tag{2.71}$$

$$\frac{\partial\omega}{\partial k_i}=\frac{f_i'-f_i'-k_i f_i''}{f_i'}-\frac{(f_i-k_i f_i')f_i''}{(f_i')^2}=\frac{-f_i f_i''}{(f_i')^2} \quad (i=1,2) \tag{2.72}$$

Thus we have (2.68):

$$\frac{\partial k_i}{\partial\omega}=\frac{-(f_i')^2}{f_i f_i''}>0 \quad (i=1,2) \tag{2.73}$$

Derivation of equation (2.69): From (2.66),

$$P=\frac{P_i}{P_j}=\frac{f_j'}{f_i'} \quad (i=1,2,\ j\neq i). \tag{2.74}$$

Thus

$$\frac{\partial P}{\partial\omega}=\left(f_j''\frac{\partial k_j}{\partial\omega}f_i'-f_j'f_i''\frac{\partial k_i}{\partial\omega}\right)\Big/(f_i')^2 \quad (i=1,2,\ j\neq i) \tag{2.75}$$

$$=\left(\frac{-(f_j')^2}{f_j}\cdot f_i'+\frac{f_j'(f_i')^2}{f_i}\right)\Big/(f_i')^2 \quad (i=1,2,\ j\neq i) \tag{2.76}$$

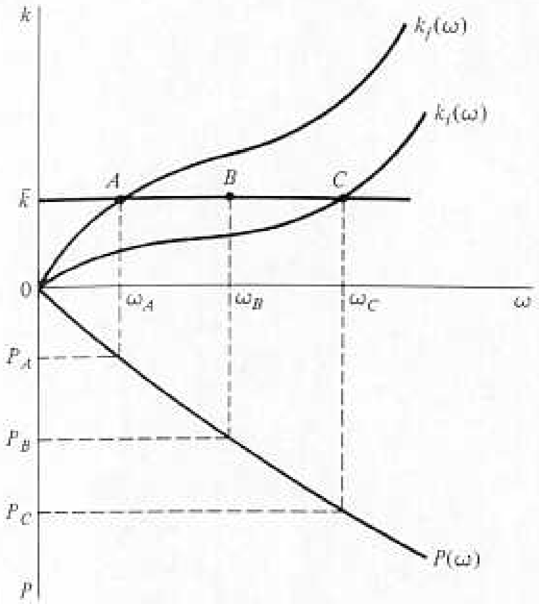

Figure 2.6. Equilibrium in a two-sector model ($P = P_i/P_j$).

$$= \left[\left(\frac{-f_j'}{f_j}\right) + \left(\frac{f_i'}{f_i}\right)\right] \cdot \frac{f_j}{f_i'} \quad (i = 1, 2,\ j \neq i). \tag{2.77}$$

Thus, since $P = f_j/f_i'$, $f_i'/f_i = 1/(k_i + \omega)$, and $f_j'/f_j = 1/(k_j + \omega)$,

$$\frac{\partial P}{\partial \omega} = P\left(\frac{1}{k_i + \omega} - \frac{1}{k_j + \omega}\right) \quad (i = 1, 2,\ j \neq i), \tag{2.78}$$

which yields (2.69):

$$\frac{1}{P} \cdot \frac{\partial P}{\partial \omega} = \left(\frac{1}{k_i + \omega} - \frac{1}{k_j + \omega}\right) \quad (i = 1, 2,\ j \neq i). \tag{2.79}$$

The manner in which competitive equilibria are characterized by this model can be displayed using (2.68) and (2.69). In Figure 2.6, $k_i(\omega)$ and $k_j(\omega)$ reflect the cost-minimizing capital intensities of the two sectors for any wage–rentals ratio ω. From (2.68), these intensities are upward sloping. Since $k_j(\omega) > k_i(\omega)$ for all values of ω, sector j is the capital-intensive sector. The ratio P_i/P_j is increasing in ω owing to (2.69), since j is capital intensive.

The economywide capital–labor ratio $\bar{k}$ is given by the initial factor endowments $\bar{K}$ and $\bar{L}$, and full employment of both factors implies that

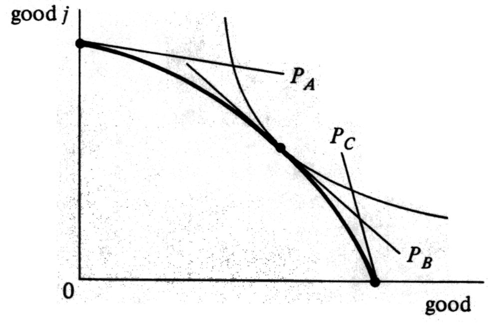

Figure 2.7. Production possibility frontier corresponding to Figure 2.6.

$$\bar{k} = \frac{K_i}{L_i} \cdot \frac{L_i}{\bar{L}} + \frac{K_j}{L_j} \cdot \frac{L_j}{\bar{L}} \tag{2.80}$$

or

$$\bar{k} = k_i l_i + k_j l_j, \quad \text{where } l_i + l_j = 1. \tag{2.81}$$

The terms l_i and l_j define the shares of labor in each of the two sectors.

Equation (2.81) thus defines a region of potential equilibrium values of ω, and through (2.69) a region of potential equilibrium values for P. At the wage–rentals ratio ω_A the cost-minimizing capital–labor ratio in sector j equals $\bar{k}$, while in sector i it is below $\bar{k}$. Thus, for (2.81) to hold, the economy must be completely specialized in the production of good j. By a similar argument, at the wage–rentals ratio ω_C, the economy will be completely specialized in the production of good i. The line segment AC thus defines a range of potential equilibria as the economy moves from complete specialization in j to complete specialization in i. The corresponding price ratios P_A and P_C define the slopes of the production possibility frontier at its terminal points, as shown in Figure 2.7. The particular equilibrium in this range that actually occurs will depend on demand conditions, which need to be specified to complete the model. For a simple one-consumer economy, such an equilibrium is depicted as point B in Figures 2.6 and 2.7. Figure 2.8 depicts factor price equalization occurring in a two-country, two-sector model.

In applying the two-sector model to the analysis of policy issues, the ability to reduce the effective dimensionality of the model when solving it is also important. This enables the two-sector model of production and exchange to be converted to a pure exchange model in factor space, which in turn makes the model easier to solve, as outlined in Figure 2.9.

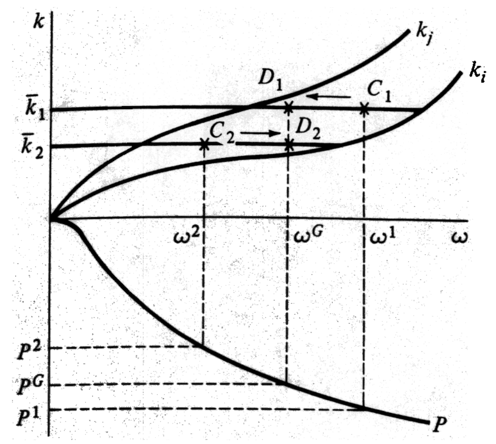

Figure 2.8. Factor price equalization in a two-country trade model.

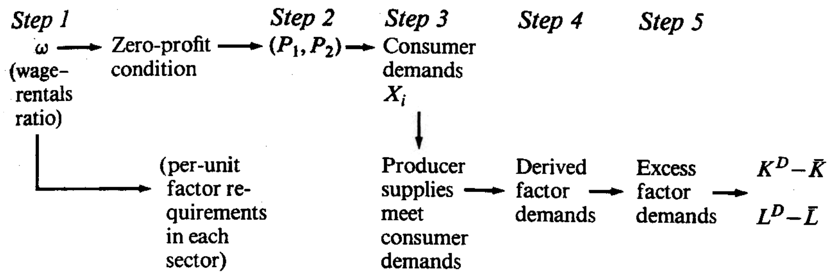

Figure 2.9. Schematic outline of dimension-reducing techniques used in solving two-sector models.

This dimension reduction works as follows. The basic two-factor, two-good model has four prices (P_L, P_K, P_1, P_2). However, an equilibrium may be characterized by (P_L, P_K) only, and the model solved using a one-dimensional unit simplex. Using the zero-profit conditions for each sector, cost-covering commodity prices (P_1, P_2) can be calculated (step 2). This enables commodity demands X_i to be calculated from consumer utility maximization (step 3). If production in each sector is set to meet commodity demands, then derived factor demand functions can be calculated (step 4), and excess factor demands determined (step 5). Since Walras's law applies to the commodity demand functions, and since the calculation of commodity prices directly imposes zero-profit conditions on the model solution, Walras's law must also apply to the excess factor demands. Because of this, and because zero-profit conditions and demand–supply equalities in goods markets are directly imposed, an equilibrium in this model can be characterized by factor prices (P_L^*, P_K^*) so that the derived factor excess demands are zero; that is,

$$K^D(P_L^*, P_K^*) - \bar{K} = 0 \quad \text{and} \quad L^D(P_L^*, P_K^*) - \bar{L} = 0. \tag{2.82}$$

If these conditions are met then an equilibrium will have been determined for the whole model, including goods markets and zero-profit conditions. As shown in Chapter 3, this same approach can also be used for goods and factors models with more than two sectors.

2.6 The normative content of general equilibrium analysis

General equilibrium analysis is widely used in modern economics in large part because it provides a wide-ranging framework that captures interactions between markets in economies. It is important, however, to note that equilibrium analysis also has a strong normative content.

This normative content is reflected in the two fundamental theorems of welfare economics (Arrow 1951), which state that any competitive equilibrium is Pareto optimal and that any Pareto optimal allocation can be supported as a competitive equilibrium with appropriate lump-sum transfers. The implication of these two theorems is that government intervention in the economy that distorts relative commodity prices will have a social cost when analyzed using a general equilibrium model. Policies such as taxes or tariffs will move the economy away from a Pareto optimal allocation and will cause a deadweight loss. In trade models this can still be nationally beneficial owing to a terms-of-trade improvement, but will remain costly from a global point of view. These theorems also imply that concerns over income-distribution effects of policy interventions should be separated from efficiency concerns; redistribution in kind, which distorts relative prices, should be resisted in favor of redistribution through lump-sum transfers.

The first of these theorems can be demonstrated with relative ease for the general equilibrium model, using an activity analysis specification of production presented earlier in the chapter. For any competitive equilibrium $(\mathbf{p}^*, X^*)$ there is an associated allocation of goods $\xi_i^m(\mathbf{p}^*)$ between M individuals, where the superscript m refers to individuals and the subscript i to commodities. Suppose that we consider an alternative allocation of goods ξ_i^m, which is superior to $\xi_i^m(\mathbf{p}^*)$ in the sense that at least one individual prefers the allocation ξ_i^m and no individual prefers the allocation $\xi_i^m(\mathbf{p}^*)$. If ξ_i^m is preferred to $\xi_i^m(\mathbf{p}^*)$ then

$$\sum_{i=1}^{N} p_i^* \xi_i^m \geq \sum_{i=1}^{N} p_i^* \xi_i^m(\mathbf{p}^*), \tag{2.83}$$

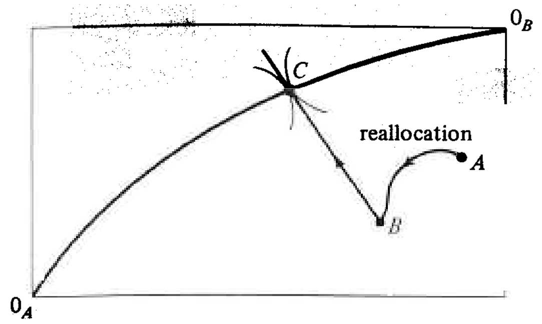

Figure 2.10. Supporting a Pareto optimal allocation as a competitive equilibrium with appropriate lump-sum transfers.

with strict inequality holding for at least one m. Thus, by Walras's law,

$$\sum_{i=1}^{N} p_i^* \xi_i > \sum_{i=1}^{N} p_i^* \xi_i(\mathbf{p}^*) = \sum_{i=1}^{N} p_i^* W_i, \tag{2.84}$$

where

$$\xi_i = \sum_{m=1}^{M} \xi_i^m \quad \text{and} \quad \xi_i(\mathbf{p}^*) = \sum_{m=1}^{M} \xi_i^m(\mathbf{p}^*). \tag{2.85}$$

This, however, is a contradiction; if ξ_i^m can be associated with $(\mathbf{p}^*, X^*)$ as an alternative general equilibrium, then multiplying both through the demand–supply equilibrium conditions by p_i^* and summing and through the zero-profit conditions by X_i^* and summing gives

$$\sum_{i=1}^{N} p_i^* \xi_i = \sum_{i=1}^{N} p_i^* W_i + \sum_{i=1}^{N} \sum_{j=1}^{K} p_i^* a_{ij} X_j^* \tag{2.86}$$

and

$$\sum_{i=1}^{N} \sum_{j=1}^{K} p_i^* a_{ij} X_j = 0. \tag{2.87}$$

However, by Walras's law,

$$\sum_{i=1}^{N} p_i^* \xi_i = \sum_{i=1}^{N} p_i^* W_i = \sum_{i=1}^{N} p_i^* \xi_i(\mathbf{p}^*), \tag{2.88}$$

which contradicts (2.84). Thus, such a general equilibrium must be Pareto optimal.

The second of the theorems follows directly, since lump-sum transfers between individuals in the pure exchange case can be used to achieve a particular competitive equilibrium outcome, even if the initial allocation of endowments is not compatible with the desired equilibrium. This is shown in Figure 2.10.

These two theorems are also usually cited as justification for reliance on the price mechanism for making resource-allocation decisions.[2] These theorems suggest that distorting policies will always have social costs when analyzed using applied general equilibrium techniques because of the deviations from conditions required for Pareto optimality. In turn, the social costs of such policies as taxes will be determined by comparing prechange and postchange equilibria, where distorting policies are replaced by allocationally neutral alternatives.

[2] However, the results due to Debreu and Scarf (1963) on the convergence to competitive equilibria of allocations in the core of an economy suggest an equivalence between political and market processes. They show that, in a pure exchange economy with a specified number of agents, increasing the number of identical agents of each type shrinks the set of core allocations, collapsing in the limit to the same allocation of resources as achieved in a competitive equilibrium. This equilibrium is not reached using any price-allocation mechanism, but relies on a procedure of proposals and blocking by coalitions.