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Notes and Comments

Are Groves–Ledyard Equilibria Attainable?

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T. Groves and J. Ledyard (1977) recently introduced a decentralized procedure for deciding upon the production and financing of public goods. Their new procedure, or mechanism, seems to avoid completely the free rider problem that most economists had believed to be unavoidable when allocating public goods. Even if individuals act only in their own self-interest, Groves and Ledyard showed, an equilibrium of their new mechanism will still be Pareto optimal. In other words, the Groves–Ledyard mechanism seems to provide a sort of invisible hand for use in allocating public goods.

When there are no externalities such as public goods, it is competition among a great many individually insignificant economic agents that enables the invisible hand of decentralized markets to achieve its beneficial results. On the other hand, when public goods are present it is commonly held that the free rider problem will be more severe the larger the economy (see, for example Olson (1965), Muench (1972), Roberts (1977), Samuelson (1969 and 1968)). It is therefore crucial to inquire of any proposed solution to the free rider problem whether it remains a solution even when there are many individuals in the economy.

In this note we present an example which suggests that there may indeed be serious difficulties with the Groves–Ledyard mechanism if it is applied to an economy in which there are many individuals. The difficulties that we identify do not concern the existence or optimality of equilibrium: Groves and Ledyard have demonstrated, (1977 and 1980), that Cournot–Nash equilibria of their mechanism will exist and will be Pareto optimal under fairly general conditions, and their results are independent of the economy's size. What our example strongly suggests, however, is that it will be impossible to actually *attain* an equilibrium under the Groves–Ledyard mechanism if the economy is very large.

The example consists of a sequence of economies in which the nth term is an economy consisting of n individuals. In this sequence of ever-larger economies, attainability of equilibria becomes asymptotically impossible in one of two ways (depending upon the exact parametrization of the Groves–Ledyard mechanism): either the mechanism grows "infinitely unstable", or else the participants' incentives to maximize become negligible.

It is important to emphasize that the equilibration failures exhibited by our example do not occur when the Walrasian "competitive" mechanism is applied to an economy with only private goods. The example therefore suggests that, as Samuelson has often maintained (Samuelson (1954, 1968 and 1969)), it may be impossible after all to duplicate, for public goods, the success of "self-policing" decentralized markets in computing a Pareto optimal equilibrium.

We now present the example, in which, for each positive integer n, we have an economy E_n with the following characteristics. There are n individuals, indexed $i =$

$1, \ldots, n$. There are two goods; x denotes the quantity of the public good that is provided for all; and y_i denotes the quantity of the private good that is allocated to individual i. Every individual makes his choices according to a utility function of the form $u_i(x, y_i) = y_i - \frac{1}{2}(x - \alpha_i)^2$. The economy has no endowment of the public good; each individual i has an endowment of ω_i units of the private good; and the public good can be produced at a constant per-capita cost of one private-good unit per person for each public-good unit—i.e. feasible allocations $(x; y_1, \ldots, y_n)$ must satisfy $nx + \sum_1^n y_i = \sum_1^n \omega_i$. The parameters α_i and ω_i are drawn from bounded intervals.

In the Groves–Ledyard mechanism each individual chooses a "message" $m_i \in \mathbb{R}$; the allocation determined by the list $(m_1, \ldots, m_n)$ of messages is

$$x = \sum_1^n m_i \quad \text{and} \quad y_i = \omega_i - x + \tfrac{1}{2}\gamma\left[\sigma_{\sim i}^2 - \frac{n-1}{n}(m_i - \bar{m}_{\sim i})^2\right], \qquad i = 1, \ldots, n,$$

where γ is a (positive) parameter of the mechanism, $m_{\sim i}$ is the $(n-1)$-tuple $(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n)$, and $\bar{m}_{\sim i}$ and $\sigma_{\sim i}^2$ are the "sample mean" and "sample variance" of $m_{\sim i}$:

$$\bar{m}_{\sim i} = \frac{1}{n-1}\sum_{j \neq i} m_j \quad \text{and} \quad \sigma_{\sim i}^2 = \frac{1}{n-2}\sum_{j \neq i}(m_j - \bar{m}_{\sim i})^2.$$

Suppose that this mechanism is implemented in any one of the economies E_n of our sequence, and suppose that it is used iteratively in that economy in an attempt to reach an equilibrium. We will examine the consequences of "Cournot" reactions by each of the participants: *viz.* if $m_i(t)$ denotes the message of individual i at iteration t and $S_i(t)$ denotes the sum $\sum_{j \neq i} m_j(t)$, we assume that $m_i(t+1)$ is chosen to maximize u_i, subject to the constraint (i's "expectation") that $S_i(t+1) = S_i(t)$.[1]

It is easy to verify that i's maximizing message is given by

$$m_i(t+1) = a_n S_i(t) + b_n(\alpha_i - 1),$$

where

$$a_n = \frac{\frac{1}{n} - \frac{1}{\gamma}}{1 + \frac{1}{\gamma} - \frac{1}{n}} \quad \text{and} \quad b_n = \frac{\frac{1}{\gamma}}{1 + \frac{1}{\gamma} - \frac{1}{n}}.$$

Thus,

$$x(t+1) = (n-1)a_n x(t) + b_n\left(\sum_1^n \alpha_i - n\right),$$

from which it follows that the Cournot-Nash equilibrium value of x is

$$x^e = \left(\sum_1^n \alpha_i - n\right)/n = \bar{\alpha} - 1,$$

and that

$$x(t+1) - x^e = (n-1)a_n[x(t) - x^e]. \tag{1}$$

As n grows large, it is clear from (1) that the mechanism will become unstable. Indeed, the mechanism will eventually overreact (because a_n is eventually negative) more and more violently the larger the economy to which it is applied.

There is, however, a way to avoid this instability. There is no reason why the parameter γ must be assigned the same value in economies of all sizes. If we instead always assign to γ a value at least as large as the number of individuals in the economy, then it is easy to verify that $0 \leqq (n-1)a_n < 1$, and (1) therefore indicates that the public good level will converge to its equilibrium value. (Indeed, if we always set γ exactly *equal* to n, then the example will always converge to x^e at the first iteration).

But this apparent convergence is misleading. When γ is quite large, the penalty attached to the term $m_i - \bar{n}_{\sim i}$ is so severe that each individual's optimal message, say $\hat{m}_i$, is virtually identical to $\bar{m}_{\sim i}$, and the resulting outcome, say $(\hat{x}, \hat{y}_i)$, for each individual—no matter what his α_i value—is virtually identical to the outcome, say $(\bar{x}, \bar{y}_i)$, had he simply chosen $m_i = \bar{m}_{\sim i}$ instead of $m_i = \hat{m}_i$. In other words, if the determination of one's own α_i-value entails any cost (e.g. in evaluating the public good's likely effect upon one), then a very large γ will lead each individual to choose $m_i = \bar{m}_{\sim i}$ in preference to incurring the cost required to choose $m_i = \hat{m}_i$.

To make the argument in the preceding paragraph precise, we consider a fixed $S_i = \sum_{j \neq 1} m_j$ and show that both $\hat{x} - \bar{x}$ and $\hat{y}_i - \bar{y}_i$, as defined there, vanish as $\gamma \to \infty$ (the result will be uniform in α_i, because of our restriction of the α_i to a bounded interval). We have $\hat{x} - \bar{x} = \hat{m}_i - \bar{m}_{\sim i}$ and

$$\hat{y}_i - \bar{y}_i = \bar{x} - \hat{x} + \tfrac{1}{2}\gamma(\hat{m}_i - \bar{m}_{\sim i})^2(n-1)/n,$$

so it will suffice to show that $(\hat{m}_i - \bar{m}_{\sim i})\sqrt{\gamma} \to 0$ as $\gamma \to \infty$. From the definitions of $\hat{m}_i$ and $\bar{m}_{\sim i}$ we have

$$\hat{m}_i - \bar{m}_{\sim i} = [a_n - 1/(n-1)]S_i + b_n(\alpha_i - 1).$$

It is clear from the definition of b_n that $b_n\sqrt{\gamma} \to 0$; moreover,

$$a_n - \frac{1}{n-1} = \frac{-n}{\gamma(n-1)(n-1) + n/\gamma},$$

from which it follows that $[a_n - 1/(n-1)]\sqrt{\gamma} \to 0$, completing the argument.

But now it is evident that we are on the horns of a dilemma. In order to avoid violent instability in large economies, γ must increase at the rate of n (precisely, γ/n must remain bounded away from zero simply to ensure that $(n-1)a_n$ does not become infinitely negative); but when n is large, such large values of γ will destroy the very incentive to optimize upon which the mechanism's optimality properties depend.[2] In short, the mechanism's Cournot–Nash equilibria appear to be irrelevant in very large economies, because there is no way to attain them.

Finally, it should be pointed out that the special features of the example (most notably, the especially simple one-parameter class of utilities) were employed to make the argument concise and transparent. Qualitatively identical results are obtained for an extremely general class of economies in Muench and Walker (1981), which also includes an argument which indicates that the results ought to be valid for a much broader class of disequilibrium behaviour than the "Cournot" behaviour posited here.

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NOTES

1. Other kinds of reaction behaviour are discussed in Muench and Walker (1981).
2. The incentive-strength argument in the preceding paragraph can be retraced without change when γ depends upon n.

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