Continuous Functions in Metric Spaces

Throughout this section let (X, d_X) and (Y, d_Y) be metric spaces.

Definition: Let $\overline{x} \in X$. A function $f : X \to Y$ is continuous at $\overline{x}$ if for every sequence $\{x_n\}$ that converges to $\overline{x}$, the sequence $\{f(x_n)\}$ converges to $f(\overline{x})$.

Definition: A function $f : X \to Y$ is continuous if it is continuous at every point in X.

Theorem: A function $f : X \to Y$ is continuous at $\overline{x}$ if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d_X(x, \overline{x}) < \delta \Rightarrow d_Y(f(x), f(\overline{x})) < \epsilon$ — *i.e.*,

$$\forall \epsilon > 0 : \exists \delta > 0 : x \in B(\overline{x}, \delta) \Rightarrow f(x) \in B(f(\overline{x}), \epsilon). \qquad (*)$$

Proof:

($\Rightarrow$:) Let $\epsilon > 0$. Suppose, by way of contradiction, that there is no $\delta > 0$ such that $d_X(x, \overline{x}) < \delta \Rightarrow d_Y(f(x), f(\overline{x})) < \epsilon$ — *i.e.*,

$$\forall \delta > 0 : \exists x \in B(\overline{x}, \delta) \text{ for which } f(x) \notin B(f(\overline{x}), \epsilon).$$

Then, in particular, for every $n \in \mathbb{N}$, let $\frac{1}{n}$ play the role of δ above: there is an $x_n \in B(\overline{x}, \frac{1}{n})$ for which $f(x_n) \notin B(f(\overline{x}), \epsilon)$. We therefore have a sequence $\{x_n\}$ in X that converges to $\overline{x}$ but the sequence $\{f(x_n)\}$ does not converge to $f(\overline{x})$, contradicting our assumption that f is continuous.

($\Leftarrow$:) Assume that $(*)$ holds, and let $\{x_n\}$ be a sequence that converges to $\overline{x}$. In order to show that $\{f(x_n)\}$ converges to $f(\overline{x})$, let $\epsilon > 0$. According to $(*)$, there is a $\delta > 0$ for which

$$x \in B(\overline{x}, \delta) \Rightarrow f(x) \in B(f(\overline{x}), \epsilon).$$

Since $\{x_n\} \to \overline{x}$, we can choose $\overline{n} \in \mathbb{N}$ such that $n > \overline{n} \Rightarrow x_n \in B(\overline{x}, \delta)$. But then

$$n > \overline{n} \Rightarrow f(x_n) \in B(f(\overline{x}), \epsilon);$$

i.e., $\{f(x_n)\}$ converges to $f(\overline{x})$, and f is therefore continuous at $\overline{x}$. □

Remark: For functions f from $\mathbb{R}^n$ to $\mathbb{R}^m$ this theorem says that f is continuous at $\overline{x} \in \mathbb{R}^n$ if and only if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|x - \overline{x}\| < \delta \Rightarrow \|f(x) - f(\overline{x})\| < \epsilon$.

Theorem: A function $f : X \to Y$ is continuous if and only if for every open set V in Y the inverse image $f^{-1}(V)$ is an open set in X.

Proof: Exercise.

An elementary consequence of the preceding theorem is its analogue in terms of closed sets:

Theorem: A function $f : X \to Y$ is continuous if and only if for every closed set S in Y the inverse image $f^{-1}(S)$ is a closed set in X.

This gives us four equivalent definitions of a continuous function f from X to Y:

If for every sequence $\{x_n\}$ that converges to $\overline{x}$, the sequence $\{f(x_n)\}$ converges to $f(\overline{x})$.

If for every $\overline{x} \in X : \forall \epsilon > 0 : \exists \delta > 0 : x \in B(\overline{x}, \delta) \Rightarrow f(x) \in B(f(\overline{x}), \epsilon)$.

If the inverse image of any open set in Y is an open set in X.

If the inverse image of any closed set in Y is a closed set in X.

Remark: We've already seen applications of these ideas to preferences and utility functions, and to the possibility of representing a preference by a utility function.

Remark: When the target space Y is actually a normed vector space, it's natural to define the sum and scalar multiple of continuous functions pointwise — *i.e.*, the functions $f + g : X \to Y$ and $\alpha f : X \to Y$ are defined by $\forall x \in X : (f + g)(x) = f(x) + g(x)$ and $\forall x \in X : (\alpha f)(x) = \alpha f(x)$. Then the set $C(X; Y)$ of all continuous functions on X into Y, with these definitions of addition and scalar multiplication, is a vector space.

Proof: Exercise. This requires showing that $C(X; Y)$ is "closed under vector addition and scalar multiplication." This does not mean that $C(X; Y)$ is a closed set, but rather that if f and g are in $C(X; Y)$ and $\alpha \in \mathbb{R}$, then $f + g$ and αf are in $C(X; Y)$ — *i.e.*, that the sum of continuous functions is a continuous function, and that a multiple of a continuous function is a continuous function.

For real-valued functions (*i.e.*, if $Y = \mathbb{R}$), we can also define the product fg and (if $\forall x \in X : f(x) \neq 0$) the reciprocal $1/f$ of functions pointwise, and we can show that if f and g are continuous then so are fg and $1/f$.

Remark: If X, Y, and Z are metric spaces, and if $f : X \to Y$ and $g : Y \to Z$ are continuous, then the composition $f \circ g : X \to Z$ is continuous.

In Euclidean space (*i.e.*, $\mathbb{R}^n$ with any norm) we say that a set is **compact** if it's both closed and bounded. One of the most important properties of continuous functions is that they "preserve" compactness — *i.e.*, if X is a compact subset of $\mathbb{R}^n$ and if $f : X \to \mathbb{R}^m$ is a continuous function, then the image of X, $f(X)$, is a compact set in $\mathbb{R}^m$. This is the Weierstrass Theorem. In fact, the Weierstrass Theorem holds in general metric spaces:

Weierstrass Theorem: If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is a compact subset of Y.

Corollary: If $f : X \to \mathbb{R}$ is a continuous real-valued function on a compact set, then f attains a maximum and a minimum on X.

Instead of proving the Weierstrass Theorem here, we defer the proof until after we've developed our next important concept, the Bolzano-Weierstrass (B-W) Property. There are two good reasons for waiting until then to do the proof: (1) we need the B-W Property in order to generalize the notion of a compact set to general metric spaces, and (2) the theorem's proof is *much* easier using the B-W Property in the general setting than if we were to do it using the closed-and-bounded definition of compactness in Euclidean space.