Differentiable Quasiconcave Functions

The original Kuhn-Tucker Theorem was stated and proved (by Harold Kuhn and Albert Tucker) for concave objective functions and convex constraint functions. But concavity and convexity are sometimes stronger properties than we want to assume for the functions we're working with. The classical example is utility functions. For example, we've already seen that the Cobb-Douglas utility function $u(\mathbf{x}) = x_1x_2$ on $\mathbb{R}^2_+$ is not concave. But it's nevertheless a "good" utility function: it has decreasing MRS everywhere — its indifference curves are all convex, *i.e.*, its upper-contour sets are all convex sets. So it's a *quasiconcave* function. We don't want to assume utility functions are concave, because the only properties of utility functions that matter are the properties of their level curves, not the actual numbers a function assigns to the vectors in its domain. But we're generally willing to assume utility functions are quasiconcave.

In order to obtain derivative conditions on a function $f : \mathbb{R}^n \to \mathbb{R}$ that are necessary or sufficient for f to be quasiconcave, let's look at Figure 1, where we have one of the level curves of a quasiconcave function f. Notice two things about Figure 1:

(i) For every nonzero $\Delta\mathbf{x} = \mathbf{x} - \overline{\mathbf{x}}$ that satisfies $\Delta f(\Delta\mathbf{x}) > 0$ — *i.e.*, $f(\mathbf{x}) > f(\overline{\mathbf{x}})$ — we have $\nabla f(\overline{\mathbf{x}})\Delta\mathbf{x} > 0$. Equivalently, $\nabla f(\overline{\mathbf{x}})\Delta\mathbf{x} \leqq 0 \Rightarrow \Delta f(\Delta\mathbf{x}) \leqq 0$. This is true *globally*, for *every* nonzero $\Delta\mathbf{x}$, not just locally, for "small" $\Delta\mathbf{x}$. Moreover, this would be true at any other $\overline{\mathbf{x}}$ on the level curve — and therefore for any $\overline{\mathbf{x}}$ in $\mathbb{R}^n$.

(ii) The diagram looks just like it would if $\overline{x}$ were a maximum of f subject to the constraint $f_1x_1 + f_2x_2 = f_1\overline{x}_1 + f_2\overline{x}_2$, where $f_i = \dfrac{\partial f}{\partial x_i}(\overline{\mathbf{x}})$ — *i.e.*, to $f_1\Delta x_1 + f_2\Delta x_2 = 0$,

Property (ii) suggests that we may be able to translate the necessary and/or sufficient conditions for a maximum point subject to constraint into conditions for f to be quasiconcave. This is almost true: indeed we *can* translate the constrained maximization conditions into conditions for Property (i) to hold, but Property (i) isn't exactly the same as quasiconcavity. (Functions with Property (i) are called *pseudoconcave*, but we're generally interested in the slightly weaker property of quasi-concavity.) However, if in addition to (i), f also has no critical points (for example, if f is strictly increasing in each of its components, like the utility function u above), then we do have the following characterization of differentiable quasiconcave functions.

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function that satisfies $\nabla f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then f is quasiconcave if and only if

$$\forall \mathbf{x}, \overline{\mathbf{x}} \in \mathbb{R}^n : \; \nabla f(\overline{\mathbf{x}})\Delta\mathbf{x} \leqq 0 \Rightarrow \Delta f(\Delta\mathbf{x}) \leqq 0.$$

This gives us the following theorem, where we exploit the fact that in the constrained maximization theorem, with a single constraint $G(\mathbf{x}) = b$, we have $\nabla f = \lambda \nabla G$ at $\overline{\mathbf{x}}$ for some nonzero λ. We simply replace $\nabla G(\overline{\mathbf{x}})$ — the top and left borders in the determinantal conditions for a constrained maximum point — with $\nabla f(\overline{\mathbf{x}})$. (We dispense with the λ by multiplying both the top and left borders by λ, which doesn't change the sign of the determinant or of any of its bordered minors.)

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function and for each $\mathbf{x} \in \mathbb{R}^n$ define the bordered Hessian $\mathbb{B}(\mathbf{x})$ as follows:

$$\mathbb{B}(\mathbf{x}) = \begin{bmatrix} 0 & f_1 & \cdots & f_n \\ f_1 & f_{11} & \cdots & f_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{bmatrix},$$

where f_i denotes the derivative $\dfrac{\partial f}{\partial x_i}(\mathbf{x})$ and f_{ij} denotes the derivative $\dfrac{\partial^2 f}{\partial x_i\,\partial x_j}(\mathbf{x})$. If $\nabla f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$, then

(i) a sufficient condition for f to be quasiconcave is $\forall \mathbf{x} \in \mathbb{R}^n\colon\ (-1)^r \left|\mathbb{B}^r(\mathbf{x})\right| > 0,\ r = 2, \ldots, n,$
for all leading principal minors $\left|\mathbb{B}^r(\mathbf{x})\right|$;

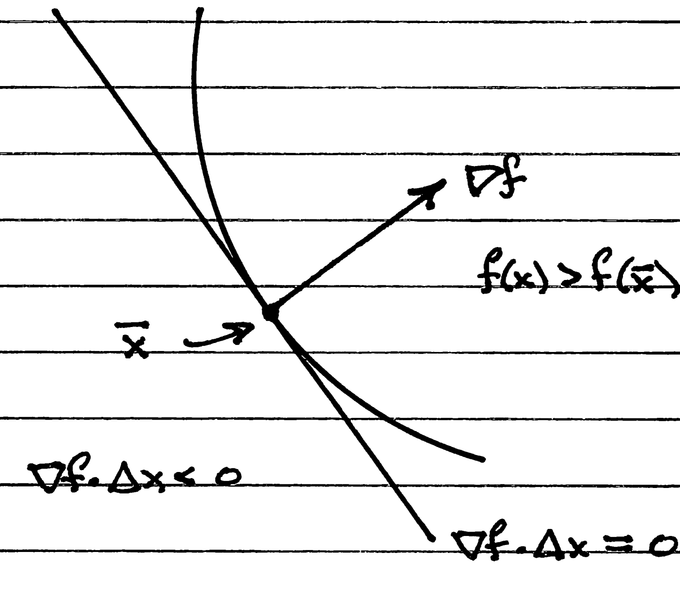
(ii) a sufficient condition for f to be quasiconvex is $\forall \mathbf{x} \in \mathbb{R}^n\colon\ \left|\mathbb{B}^r(\mathbf{x})\right| < 0,\ r = 2, \ldots, n,$
for all leading principal minors $\left|\mathbb{B}^r(\mathbf{x})\right|$;

(iii) a necessary condition for f to be quasiconcave is $\forall \mathbf{x} \in \mathbb{R}^n\colon\ (-1)^r \left|\mathbb{B}^r(\mathbf{x})\right| \geqq 0,\ r = 2, \ldots, n,$
for *all* border-preserving principal minors $\left|\mathbb{B}^r(\mathbf{x})\right|$;

(iv) a necessary condition for f to be quasiconvex is $\forall \mathbf{x} \in \mathbb{R}^n\colon\ \left|\mathbb{B}^r(\mathbf{x})\right| \leqq 0,\ r = 2, \ldots, n,$
for *all* border-preserving principal minors $\left|\mathbb{B}^r(\mathbf{x})\right|$.

You should compare these conditions to the conditions for definiteness of quadratic forms subject to homogeneous linear constraints, for just one constraint, as described in the paragraph preceding the theorem.

FIGURE 1:



$f(x) \leqq f(\bar{x})$

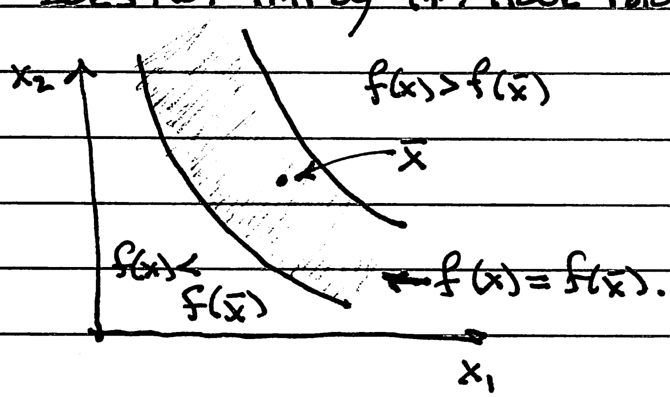
(*) $\forall \Delta x \neq \underline{0}: \ \nabla f(\bar{x}) \cdot \Delta x \leqq 0 \Rightarrow \Delta f(\Delta x) \leqq 0$

IF (*) IS TRUE FOR ALL $\bar{x} \in \text{dom} f$, THEN f IS QUASICONCAVE.

(THE CONVERSE IS NOT QUITE TRUE, BUT IT IS IF $\nabla f(\bar{x}) \neq \underline{0}$ FOR ALL $\bar{x} \in \text{dom} f$.)

[(*) IS CALLED PSEUDOCONCAVITY]

COUNTEREXAMPLE TO SHOW THAT f QUASICONCAVE DOES NOT IMPLY (*) TRUE FOR ALL $\bar{x} \in \text{dom} f$:



f IS QUASICONCAVE;

$\nabla f(\bar{x}) = \underline{0}$;

$\therefore \nabla f(\bar{x}) \cdot \Delta x = 0, \ \forall \Delta x \in \mathbb{R}^2$

~~BUT WE DON'T HAVE~~

~~$f(x) \leqq f(\bar{x})$ FOR~~

ALL $\bar{x} \in \text{dom} f$,

i.e., FOR ALL Δx.