Example: Linear Programming

A linear programming problem is a nonlinear programming problem in which all functions (objective function and constraint functions) are linear. Here's a simple linear programming problem: Suppose a firm produces two products and uses three inputs in the production process. The firm already has on hand 70 units of input #1, 40 units of input #2, and 90 units of input #3. Production of each unit of Product #1 requires two units of input #1 and one unit of each of the other two inputs; each unit of Product #2 requires three units of input #3 and one unit of each of the other two inputs. The two products can be sold at unit prices of \$40 and \$60, respectively. How many units of each product should the firm produce in order to maximize its profit? (This is equivalent to maximizing revenue here, since the inputs are already on hand and therefore cost nothing.) In other words, what's the best way for the firm to allocate its stock of inputs to production of its two products?

The firm's problem is described analytically in the following constrained maximization problem, which is a linear programming problem. (The Lagrange multipliers σ_i associated with the constraints are not part of the LP problem, but it's useful to specify them here.)

$$\begin{aligned} \max_{(x_1,x_2)\in\mathbb{R}^2_+} \ & \pi(x_1,x_2) = 40x_1 + 60x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leqq 70 \quad : \sigma_1 \qquad (1) \\ & x_1 + x_2 \leqq 40 \quad : \sigma_2 \qquad (2) \\ & x_1 + 3x_2 \leqq 90 \quad : \sigma_3 \qquad (3) \end{aligned} \tag{LP}$$

Figure 1 depicts the three constraints; you should add two or three contours of the objective function. Note that constraints (1) and (2) intersect at the point $(30, 10)$ and that constraints (2) and (3) intersect at the point $(15, 25)$. Constraints (1) and (3) intersect at the point $(24, 22)$, which is outside the feasible set (it violates constraint (2)). It's easy to see that the profit-maximizing production plan is $(x_1, x_2) = (15, 25)$, which yields profit of $\pi = (40)(15) + (60)(25) = 2100$. (For example, it's clear from the geometry that this plan attains the highest objective contour in the feasible set.)

The first-order marginal conditions for an interior-point solution are

$$x_1 : \quad 40 = 2\sigma_1 + \sigma_2 + \sigma_3$$

$$x_2 : \quad 60 = \sigma_1 + \sigma_2 + 3\sigma_3$$

The remaining first-order conditions – the complementary slackness conditions – are

$$\sigma_1(2x_1 + x_2 - 70) = 0$$

$$\sigma_2(x_1 + x_2 - 40) = 0$$

$$\sigma_3(x_1 + 3x_2 - 90) = 0$$

We've suggested above that the plan $(15, 25)$ appears to be the solution of the maximization problem (LP). Let's check whether the first-order conditions are satisfied at the point $(15, 25)$. First note that $(15, 25)$ lies inside the first constraint, so according to the first complementary slackness condition we must have $\sigma_1 = 0$. The other two complementary slackness conditions are clearly satisfied, because $(15, 25)$ satisfies those two constraints exactly (geometrically, the point lies *on* the constraints). With $\sigma_1 = 0$ the two marginal conditions are easy to solve for σ_2 and σ_3: $\sigma_2 = 30$ and $\sigma_3 = 10$. So all five first-order conditions are satisfied at $(15, 25)$.

It's useful to notice an important consequence of the linearity of the objective function: the marginal conditions are independent of x_1 and x_2. The derivatives of a linear function are constant, independent of the variables' values; they're simply the coefficients of the variables. Therefore the marginal conditions aren't equations that can be solved to obtain the optimal values of the variables, unlike the typical situation when the objective function is not linear. In fact, there are powerful alternative techniques for solving LP problems, such as the Simplex Method, that specifically exploit the problems' complete linearity. We will not address those solution techniques in these notes. The notes focus only on the Lagrange multipliers as shadow values.

Lagrange Multipliers as Shadow Values

Now suppose the firm has thirty more units of input #3, so that constraint (3) is now

$$x_1 + 3x_2 \leqq 120.$$

Constraints (2) and (3) now intersect at the point $(0, 40)$, which is the solution of the revised LP problem. The firm's profit will now be $\pi = (40)(0) + (60)(40) = 2400$, and we therefore have

$$\Delta b_3 = 30, \quad \Delta\pi = 300, \quad \text{and} \quad \frac{\Delta\pi}{\Delta b_3} = 10,$$

where b_3 denotes the right-hand-side (RHS) coefficient in constraint (3). Thus, for sufficiently small marginal changes in the RHS of constraint (3), the resulting effect on the objective value π is given by σ_3. And this tells us that the firm should be willing to pay up to $\sigma_3 = \$10$ per unit to obtain additional units of input #3, or be willing to sell input #3 for any amount *above* $\sigma_3 = \$10$ per unit. In other words, the marginal value of input #3 to the firm is given by σ_3.

Instead of "relaxing" constraint (3), as we did above, let's *tighten* constraint (2): suppose the firm has only 30 units of input #2, so that constraint (2) is now

$$x_1 + x_2 \leqq 30.$$

Constraints (2) and (3) now intersect at the point $(0, 30)$, which is the solution of the revised LP problem. The firm's profit will now be $\pi = (40)(0) + (60)(30) = 1800$, and we therefore have

$$\Delta b_2 = -10, \quad \Delta\pi = -300, \quad \text{and } \frac{\Delta\pi}{\Delta b_2} = 30.$$

Thus, for sufficiently small marginal changes in the RHS of constraint (2), the resulting effect on the objective value π is given by σ_2. This tells us that the firm should be willing to pay up to $\sigma_2 = \$30$ per unit to obtain additional units of input #2 – or alternatively, the firm should be willing to *sell* input #2 for any amount *above* $\sigma_2 = \$30$ per unit. In other words, the marginal value of input #2 to the firm is given by σ_2.

Finally, note that the marginal value of input #1 is $\sigma_1 = 0$: constraint (1) is not binding at the optimal solution, so having a little bit more or a little bit less of input #1 will have no effect on either the optimal solution or the resulting profit.

Exercise: Determine the optimal solution if the per-unit revenue from Product #2 is \$30 instead of \$60, and determine the values of all three Lagrange multipliers.

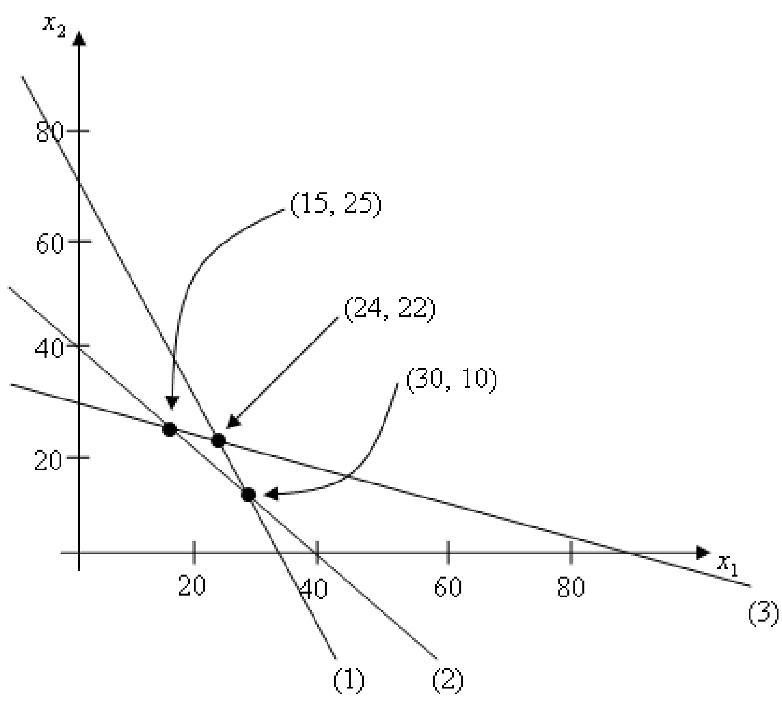

Figure 1