JOURNAL OF ECONOMIC THEORY 50, 459–464 (1990)

Maximal Elements of Weakly Continuous Relations*

DONALD E. CAMPBELL

Department of Economics, The College of William and Mary, Williamsburg, Virginia 23185, and

AND

MARK WALKER

Department of Economics, State University of New York, Stony Brook, New York 11794

Received August 7, 1987; revised June 19, 1989

A weaker than usual continuity property is defined for binary relations. Relations that have this property, along with certain transitivity properties, are shown to have maximal elements on compact sets. The results cover "interval orders," the kind of relations that often characterize choice situations in which similar alternatives are indistinguishable. *Journal of Economic Literature* Classification Numbers: 022, 213. © 1990 Academic Press, Inc.

According to the Bergstrom–Walker Theorem [1, 8], if an acyclic binary relation defined on a topological space X is lower continuous, then every compact subset of X will contain a maximal element.[1] We introduce here a much weaker continuity property for relations, and we show that if a relation is only *weakly* lower continuous and has a slightly stronger order property than acyclicity, then it will again have a maximal element on any

* Campbell's research was financed by the Social Sciences and Humanities Research Council of Canada. We are grateful for discussions with Larry Epstein.

[1] Bergstrom [1] and Walker [8]. The theorem is elementary, as the proof in [8] demonstrates. If the relation's order properties are even weaker than acyclicity, but it is defined on a convex subset of a linear topological space and its upper-contour sets are convex, then several much deeper theorems are available to ensure the presence of maximal elements; see, for example, Sonnenschein [7], Shafer and Sonnenschein [6], and Yannelis and Prabhakar [9].

642/50/2-15

0022-0531/90 $3.00

compact set. We will also show that an interval order[2] has a maximal element on compact sets if it is weakly lower continuous.

Henceforth, let X denote a topological space; let $\succ$ denote a "preference relation" (i.e., an asymmetric binary relation) on X; and let $\succsim$ denote the completion of $\succ$ (i.e., $x \succsim y$ means that $y \succ x$ does not hold). A *cycle* of length n is a finite list $(x_1, ..., x_n)$ that satisfies $x_i \succsim x_{i+1}$ for all $i \leqslant n$ (where $n+1$ is interpreted to be 1). Note that the elements of a cycle need not be distinct. A (*weak*) *link* in a cycle is a pair of elements that are adjacent in the list. We index the links by their first components: thus, the ith link is (x_i, x_{i+1}). A *strong link* is a link for which $x_i \succ x_{i+1}$. A *strong cycle* is a cycle in which each link is strong. An *alternating cycle* is a cycle whose length is even and for which each even link is strong (equivalently, each odd link is strong). A *strictly alternating cycle* is an alternating cycle in which each link consists of distinct elements.

We say that the relation $\succ$ is

(1) *Acyclic* if $x \succ y \succ z$ implies $z \succsim x$—that is, if it has no strong cycles.

(2) *Pseudotransitive* if it is acyclic and $x' \succ x \succsim y' \succ y$ implies $x' \succ y$ when $x \neq y'$.[3]

(3) *Transitive* (or weakly transitive) if $x \succ y \succ z$ implies $x \succ z$.

(4) *Extratransitive* if $x' \succ x \succsim y' \succ y$ implies $x' \succ y$ (even when $x = y'$).[3]

(5) *Fully transitive* if its completion $\succsim$ is transitive. Such relations $\succsim$ are often called *preorders*.

Each of the following remarks is easy to prove:

Remark 1. $\succ$ is extratransitive if and only if it has no alternating cycles. $\succ$ is pseudotransitive if and only if it is acyclic and has no strictly alternating cycles.

Remark 2. $\succ$ is extratransitive if and only if it is an *interval order*—i.e., if $x' \succ x$ and $y' \succ y$ together imply that $x' \succ y$ or $y' \succ x$.

Remark 3. All implications in the following diagram hold, and none of their converses holds:

[2] See Fishburn [3], Bridges [2], Gensemer [4].

[3] Bridges [2] uses the term "pseudotransitive" to refer to the property we call extratransitive.

(5) Fully tansitive

⇓

(4) Extratransitive

⇙ ⇘

(2) Pseudotransitive (3) (Weakly) Transitive

⇘ ⇙

(1) Acyclic

Remark 4. $\succ$ is extratransitive if and only if it is both transitive and pseudotransitive. In other words, (4) is the weakest condition that implies both (2) and (3).

The relation $\succ$ is said to be *lower continuous* (lc) if for each $x \in X$ the set $\{y \in X \mid x \succ y\}$ is open, and is said to be *weakly lower continuous* (wlc) if whenever $x \succ y$ there is a neighborhood of y, denoted $N(y)$, which satisfies $x \succsim N(y)$, by which notation we mean that $x \succsim z$ for every $z \in N(y)$.

For an example of a weakly lower continuous relation that fails to be lower continuous,[4] consider an individual who, for some real $\varepsilon > 0$, cannot distinguish quantities that differ by less than ε, and who therefore chooses from a set X of real numbers according to the relation $\succ$ defined by "$x \succ y$ if and only if $x \geqslant y + \varepsilon$" ("$x$ is preferred to y if and only if it is discernably larger than y"). If X has the topology it usually inherits as a set of real numbers, then $\succ$ is weakly lower continuous but not lower continuous. If X is compact, then it clearly has a maximal element, but the Bergstrom–Walker Theorem does not apply here—precisely because $\succ$ is not lower continuous ($\succ$ *is* acyclic; indeed , it is extratransitive).

It is natural to conjecture that the Bergstrom–Walker Theorem will remain true if its lower-continuity requirement is replaced by weak lower continuity. We will provide an example to show that this conjecture is false. But by strengthening the theorem's acyclicity requirement to pseudotransitivity, we obtain the following theorem, a result parallel to—if not quite stronger than—Bergstrom–Walker. (A space is T_1 if all its singleton sets are closed—in other words, if "points are closed.")

THEOREM 1. *If X is a compact T_1-space, then every weakly lower continuous pseudotransitive relation on X has a maximal element.*

[4] A second example is perhaps even more striking: Let x be any set of real numbers with a nonempty interior (in the usual topology), and let $x \succ y$ if and only if x is rational and y is irrational.

A proof of Theorem 1 will turn out to be a byproduct of the proof we will give for Theorem 2, which deals with extratransitive relations, often called interval orders. Bridges [2] has determined conditions under which an interval order on $\mathbb{R}^m$ will have a "two-function" utility representation (a maximizer of either function will then be a maximal element) and conditions under which the "utility" functions will be continuous (a maximal element will then exist on any compact set). The following theorem provides much weaker and more intuitive conditions under which an interval order will have a maximal element on compact sets.

THEOREM 2. *If X is compact, then every weakly lower continuous extratransitive relation (i.e., every wlc interval order) $\succ$ on X has a maximal element.*

Proof. Suppose that X has no maximal element. Then with each $x \in X$ we can associate both an $x' \in X$ that satisfies $x' \succ x$ and (because $\succ$ is wlc) an open set $N(x)$ containing x that satisfies $x' \succsim N(x)$. The collection of all the sets $N(x)$ is an open cover of X; because X is compact, there is a finite set $\{x_1, ..., x_n\}$ for which the collection $\{N(x_1), ..., N(x_n)\}$ covers X. Because no element of X is maximal, there is for each index i an element $y_i \in X$ that satisfies $y_i \succ x_i'$. We will show that among all the elements x_i' and y_i there must be an alternating cycle. First note that for each i, y_i is an element of one of the covering sets $N(x_j)$, but that if $y_i \in N(x_i)$, then we would have $x_i' \succsim y_i \succ x_i'$, a contradiction; thus, $y_i \notin N(x_i)$ for each i. In particular, $y_1 \notin N(x_1)$; without loss of generality, then, let $y_1 \in N(x_2)$. Now for an arbitrary i, suppose we had already shown, just as we did for $i = 1$, that $y_i \in N(x_{i+1})$; then we could not have $y_{i+1} \in N(x_j)$ for any $j \leqslant i+1$, because that would yield $x_j' \succsim y_{i+1} \succ x_{i+1}' \succsim y_i \succ x_i' \succsim \cdots \succsim y_j \succ x_j'$, an alternating cycle; without loss of generality then, we can say that $y_{i+1} \in N(x_{i+2})$, and we can say this for every i—except when $i+1 = n$, in which case we simply have $y_n \notin N(x_j)$ for all j, which we have already shown is impossible. ∎

For Theorem 1, the preceding proof fails only because it cannot establish at each step that the alternating cycle is *strictly* alternating—i.e., that in the links $x_h' \succsim y_k$ we have $x_h' \neq y_k$. Note that each such link is a direct consequence of an assumption that $y_k \in N(x_h)$. If the space X is T_1, then each of the open sets $N(x)$ can be chosen in such a way that $x' \notin N(x)$, and therefore $y_k \neq x_h'$ follows automatically from $y_k \in N(x_h)$.

In Theorem 3 we adapt Rader's technique [5] to establish that a utility representation will exist for any fully transitive relation (a relation whose completion is a preorder) that is weakly lower continuous on a second-countable space. As the example in footnote 4 demonstrates, the representation will not in general be continuous; but for the existence question alone, Theorem 3 generalizes Rader's theorem.

THEOREM 3. *If X has a countable base and $\succsim$ is a weakly lower continuous preorder on X, then there is a utility representation of $\succsim$—i.e., there is a real-valued function u on X that satisfies the condition $[x \succsim y$ if and only $u(x) \geqslant u(y)]$.*

Proof. Let $\{B_1, B_2, ...\}$ be a countable base for X. Define the real-valued functions f and g on X has follows:

$$f(x) = \sum \{2^{-n} \mid x \succ B_n\} \qquad \text{and} \qquad g(x) = \sum \{2^{-n} \mid x \succsim B_n\}.$$

We show that the function $u = f + g$ is a utility representation of $\succ$. Transitivity of $\succsim$ ensures that when $x \succsim y$, both $f(x) \geqslant f(y)$ and $g(x) \geqslant g(y)$ must hold. Thus, we need only show that when $x \succ y$, either $f(x) > f(y)$ or $g(x) > g(y)$. Suppose that $x \succ y$ and $g(x) = g(y)$. Weak lower continuity of $\succ$ ensures that $x \succsim B_n$ for some n such that $y \in B_n$, and then, because $g(x) = g(y)$, it follows that $y \succsim B_n$. Hence, because $\succsim$ is both complete and transitive, we have $x \succ B_n$. But since $y \in B_n$, we cannot have $y \succ B_n$. Consequently, $f(x) > f(y)$. ∎

In view of Theorems 1 and 2, in which pseudotransitivity and extratransitivity are each sufficient to guarantee that a wlc relation will have a maximal element on a compact set, it is natural to conjecture that (weak) transitivity of $\succ$ might also be sufficient. The following example demonstrates that this is not true.

EXAMPLE. The space X is the unit circle in the Euclidean plane, with its usual topology. The relation $\succ$ is defined as follows:

$$x \succ y \quad \text{if} \quad 0 \leqslant \arg(y) < \arg(x) < \pi \text{ or}$$
$$\pi \leqslant \arg(y) < \arg(x) < 2\pi,$$

where $\arg(z)$ denotes the angle between the point z and the horizontal axis. This relation is wlc but not lc, and is (weakly) transitive but neither extratransitive nor pseudotransitive. The space X is compact (and connected), but it contains no maximal element.

It remains an open question whether, if X were a *convex* compact subset of a linear topological space, transitivity or an even weaker order condition would suffice to ensure that a weakly lower continuous relation on X, all of whose upper-contour sets are convex, will always have a maximal element. Also open is the question whether the T_1 restriction can be dispensed with in Theorem 1.

References

1. T. C. Bergstrom, Maximal elements of acyclic relations on compact sets, *J. Econ. Theory* **10** (1975), 403–404.
2. D. S. Bridges, A numerical representation of preferences with intransitive indifference, *J. Math. Econ.* **11** (1983), 25–42.
3. P. C. Fishburn, Interval representations for interval orders and semiorders, *J. Math. Psych.* **10** (1973), 91–105.
4. S. H. Gensemer, Continuous semiorder representations, *J. Math. Econ.* **16** (1987), 275–289.
5. T. Rader, The existence of a utility function to represent preferences, *Rev. Econ. Stud.* **30** (1963), 229–232.
6. W. Shafer and H. Sonnenschein, Equilibrium in abstract economies without ordered preferences, *J. Math. Econ.* **2** (1977), 345–348.
7. H. Sonnenschein, Demand theory without transitive preferences, with an application to the theory of competitive equilibrium, *in* "Preferences, Utility, and Demand" (J. Chipman *et al.*, Eds.) Harcourt Brace Jovanovich, New York, 1971.
8. M. Walker, On the existence of maximal elements, *J. Econ. Theory* **16** (1977), 470–474.
9. N. C. Yannelis and N. Prabhakar, Existence of maximal elements and equilibria in linear topological spaces, *J. Math. Econ.* **12** (1983), 233–245.