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A Simple Auctioneerless Mechanism with Walrasian Properties*

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A simple mechanism for reallocating holdings is described, in which no auctioneer is required: outcomes are determined solely from traders' actions and without any requirement that the mechanism be in equilibrium. The mechanism is shown to exactly duplicate the performance of the Walrasian auctioneer (both in its equilibria and in its disequilibrium path) if individuals are price takers, and, if the number of individuals is large, to *approximately* duplicate the auctioneer's performance even when individuals behave strategically, each taking account of his own influence on prices. *Journal of Economic Literature* Classification Numbers: 021, 022.

In the last ten years or so, a number of research papers have adopted Cournot's "strategic" approach in the study of economic equilibrium. Instead of relying upon an auctioneer or other similar device to generate prices, it is shown in this approach that prices can be determined directly from the actions of the traders. In this paper I will present a particularly simple mechanism in which traders' actions determine prices and in which, moreover, the Walrasian tatonnement can be emulated via the maximizing actions of the traders alone, with no auctioneer required to adjust the prices.

When individual traders are price takers, this new mechanism duplicates exactly the performance of the Walrasian auctioneer: the mechanism has exactly the same set of equilibria, and it follows the same path when out of equilibrium as the Walrasian tatonnement. What is more interesting is that when there are a great many individual traders, the mechanism essentially retains its Walrasian properties, even if the individuals practice Cournot behavior instead of price-taking behavior. Specifically, the larger is the economy, the more nearly do the mechanism's Cournot–Nash equilibria coincide with the economy's Walras equilibria, and the more nearly do the

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mechanism's stability properties under Cournot best-replay behavior coincide with Walrasian stability under price-taking behavior. It seems natural, therefore, to refer to the mechanism as the *Cournot–Walras* (or briefly, the *CW*) *mechanism.*

The CW mechanism is closely related to two distinct strands of recent Cournot-style equilibrium research. The first strand begins with a paper by Shubik [22], and includes papers by Shapley [20], Shapley and Shubik [21], Pazner and Schmeidler [17], Jaynes, Okuno, and Schneidler [10], Postlewaite and Schmeidler [18], and Dubey, Mas-Colell, and Shubik [2]. The emphasis in this first strand is essentially a "positive" or descriptive one: the research consists of attempts to model the observed market process of exchange in strategic terms, and to determine whether the resulting outcomes tend to be nearly Walrasian, or at least nearly Pareto optimal, if the number of individuals involved is large.

Although the CW mechanism is similar to the models in this Shubik line of research, it is nevertheless not intended as a description or idealization of any actual, observed process of price formation. Instead, the model I will present is motivated by ideas that are much closer to the other strand of Cournot-style research, a strand that began with a paper by Hurwicz [8], and that has roots that can be traced back at least to papers by Hurwicz [28] and Farquharson [3]. The emphasis in this Hurwicz strand of research has been more normative, or comparative, than in the Shubik strand, addressing such questions as whether there exist any institutional arrangements (i.e., any "mechanism") which will yield some specified pattern of outcomes (e.g., Walrasian outcomes) when participants take advantage of their opportunities for strategic behavior; or, if there is more than one such mechanism, whether there are other performance criteria in terms of which the alternative mechanisms can be compared. Within this line of research, Schmeidler [19] and Hurwicz [9] have devised mechanisms that obtain precisely the Walrasian prices and allocations as Cournot–Nash equilibria (whether the economy is large or not). When public goods are present, similar mechanisms devised by Groves and Ledyard [6], Hurwicz [9], and Walker [25] obtain Pareto optimal, or even Lindahl, allocations as Cournot–Nash equilibria.

Unfortunately, all of the mechanisms in this Hurwicz line of research are extremely unstable: if they are not already *at* an equilibrium, there is really no hope of *finding* an equilibrium. By way of contrast, the Walrasian tatonnement, while not always stable, can nevertheless be "stabilized," or *made* stable, by judicious use of a small amount of information about the economy's parameters (see, for example, Muench [14], as well as Section 7, below). This contrast between the relative stability of the Walrasian tatonnement and the extreme instability of all the mechanisms in one strand of Cournot-style research raises the question whether the other (Shubik) strand

of Cournot-type models could be stable, and more generally, whether it is possible to duplicate the stability (as well as the equilibria) of the Walrasian tatonnement without employing an auctioneer. The CW mechanism provides a positive answer to that question.

Analysis of a mechanism's (or a game's) stability properties requires us to know something about the reactions of participants to disequilibrium situations. To put it in the simplest terms, how does a participant form his expectations about others' behavior on a given play, and how does he translate his expectations into a choice of an action on that play? There is little to guide us here, and I will therefore use the behavioral assumption that has most often been used in previous analyses of stability in noncooperative games (see, for example [1, 4, 5, 7, 12, 13, 24, 27]): each participant will be assumed to practice *best-replay behavior*. In other words, a participant will be assumed, on a given play, to choose an action that would be utility-maximizing for him *if* all data that he receives concerning others' actions were to remain unchanged from the previous play.

Arrow and Hurwicz [1], Williams [27], Gabay and Moulin [5], and Luenberger [12] have all given conditions under which a noncooperative game will be stable if all players practice Cournot best-replay behavior; however, none of the economic games in either the Shubik or Hurwicz strands of research are covered by these conditions. Indeed, all of the mechanisms in the Hurwicz strand exhibit severe *in*stability under Cournot best-replay behavior (this is quite easy to see for the Hurwicz and Schmeidler mechanisms; for the other two, see [15, 16, 26]). It is therefore interesting to discover, as we will do below, that a very simple mechanism of the Shubik type will not only be stable under Cournot best-replay behavior, but that it will in fact duplicate the Walrasian tatonnement.

The remainder of the paper is organized into eight sections, as follows. 1. The CW Mechanism; 2. Individual Behavior; 3. Price-Taking Behavior; 4. Cournot Behavior; 5. Definitions; 6. Cournot Equilibrium; 7. Stability under Cournot Behavior; 8. Concluding Remarks. The analysis is carried out within a very simple framework: there are only two goods and there is no production; all preferences are smooth and only interior equilibria are considered; and "large economies" are analyzed via sequences of ever-larger finite economies, so that an infinite limit economy (and its accompanying mathematical machinery) is not required. And, of course, there is the restriction to best-replay behavior that I have described above.

1. The CW Mechanism

Suppose that there are n individuals, or *players*, indexed $i = 1,..., n$, among whom the two goods are to be reallocated. Quantities of the goods will be

denoted by x and y. The CW mechanism is defined by the requirement that each player announce a real number m_i, called his *message*, and by the following *outcome functions*, which determine each player's *net trade* (x_i, y_i) as a function of the n-tuple $m = (m_1,..., m_n)$ of messages that are announced:

$$X_i(m) = m_i - \bar{m} \qquad \text{and} \qquad Y_i(m) = -\gamma\bar{m}X_i(m), \tag{1.1}$$

where γ is a positive number and $\bar{m}$ is the mean of all n messages:

$$\bar{m} = \frac{1}{n}\sum_{j=1}^{n} m_j.$$

The number $\gamma\bar{m}$ can be interpreted as the price per unit of X (expressed in units of Y) that every buyer of X pays and every seller of X receives. Each player's choice of m_i will of course have a direct influence on this price, but the influence will be extremely small when n is large. We will therefore investigate the consequences, for both equilibrium and stability, of two kinds of behavior:

(a) Under "price-taking" behavior, in which players treat $\bar{m}$ as parametric, we will find that the equilibria yield exactly Walrasian allocations and that $\gamma\bar{m}$ will be an exact Walrasian price, and (if players expect $\bar{m}$ at the current play to be the same as at the previous play) the disequilibrium adjustment path will be exactly the path that the Walrasian tatonnement would follow if the Walrasian "adjustment speed" were γ.

(b) Under "Cournot" behavior, in which each player i treats the mean $\bar{m}_{\sim i} = [1/(n-1)]\sum_{j\neq i} m_j$ of all *other* players' messages as parametric, we will find that if n is large then the results in (a) will not be exactly true but will be *approximately* true (where "large" and "approximate" will be made precise in a sequential framework).

2. Individual Behavior

In order to avoid confusion about terminology, it will be well to clarify the way that the three adjectives "price-taking," "Cournot," and "best-replay" are being used. A player's behavior is said to be *best-replay* if, whether the game is in equilibrium or not, the player's action at play $t+1$ is utility-maximizing under the assumption that some specified data will be unchanged at $t+1$ from what they were at play t. On the other hand, to say that a player's behavior is *price-taking* (or, respectively, *Cournot*) is to tell us what data it is that the player is treating as parametric, i.e., which data he believes will be unaffected by his own (current) action: the "price" $\gamma\bar{m}$ is treated as parametric in price-taking behavior, and the other $n-1$ players' messages in

Cournot behavior. (Throughout this paper, the term "Cournot behavior" can be interpreted to have the much weaker meaning that a player, say i, treats only the *mean* $\bar{m}_{\sim i}$ of the others' messages as parametric.)

When we *combine* the adjectives—"price-taking best-replay behavior" or "Cournot best-replay behavior"—then we know not only which data are taken to be parameters in a player's maximization problem, but also how he determines the values of the parameters: by adopting the most recent observed values of them.

3. Price-Taking Behavior

Let $D_i(\cdot)$ denote player i's Walrasian demand function for the X-good (net of his initial holdings). A price-taking player, when faced with the price $\gamma\bar{m}$, would simply express his desire to purchase $D_i(\gamma\bar{m})$ units. In order to express that level of demand, according to (1.1), his message m_i would have to be $m_i = D_i(\gamma\bar{m}) + \bar{m}$: he would treat $\gamma\bar{m}$ as the given price and $\bar{m}$ as a lump-sum (quantity) tax. Consequently, if *all* players were to practice price-taking behavior, then an equilibrium would satisfy $\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} D_i(\gamma\bar{m}) + n\bar{m}$—i.e., $\sum_{i=1}^{n} D_i(\gamma\bar{m}) = 0$. In other words, $\gamma\bar{m}$ would be an exact Walrasian equilibrium price, and the resulting allocation would be the corresponding Walrasian one.

If, in addition to being price-takers—i.e., each ignoring his own influence on the price—all players were also practicing best-replay behavior, then we would have

$$\gamma\bar{m}(t+1) = \gamma\bar{D}(\gamma\bar{m}(t)) + \gamma\bar{m}(t),$$

or, if we write p for $\gamma\bar{m}$,

$$p(t+1) = p(t) + \gamma\bar{D}(p(t)).$$

In other words, the economy will react to a disequilibrium by following precisely the same path that the Walrasian tatonnement would have followed if the tatonnement's "speed of adjustment" had been γ.

4. Cournot Behavior

What if a player takes account of the effect of his own message upon the per-unit price of the X-good? Qualitatively, it is clear that the more X the player demands (i.e., the larger is m_i), the more he will raise the price, and that the more X he supplies, he further he will depress the price. In order to

be more precise, we can rewrite the outcome function $Y_i(m) = -\gamma\bar{m}X_i(m)$ in the form

$$\begin{aligned} y_i &= -\gamma\left(\frac{n-1}{n}\right)\left(\frac{1}{n-1}\sum_{j=1}^{n} m_j\right)x_i \\ &= -\gamma\left[\frac{1}{n-1}\sum_{j=1}^{n} m_j - \left(\frac{1}{n}\right)\frac{1}{n-1}\sum_{j=1}^{n} m_j\right]x_i \\ &= -\gamma\left[\bar{m}_{\sim i} + \left(\frac{1}{n-1}m_i - \frac{1}{n-1}\bar{m}\right)\right]x_i \\ &= -\gamma\left[\bar{m}_{\sim i} + \frac{1}{n-1}x_i\right]x_i, \end{aligned} \tag{4.1}$$

which makes it clear that the price he must pay (if a buyer of X) increases linearly with the amount that he purchases and that the price he receives (if a seller) decreases linearly with the amount that he sells.

Geometrically, the player faces the quadratic constraint (4.1) in his space of possible trades (x_i, y_i), and the constraint is tangent at the trade $(0, 0)$ to the line $y_i = -\gamma\bar{m}_{\sim i}x_i$. This is depicted in Fig. 1.

The player's choice will of course occur at a trade that equates his marginal rate of substitution (MRS) and the slope of the constraint (4.1), which is found via differentiation of (4.1) with respect to x_i:

$$\mathrm{MRS}_i = \gamma\bar{m}_{\sim i} + \frac{2}{n-1}\gamma x_i. \tag{4.2}$$

Thus, when n is large, all players' MRSs will be nearly equal to one another (if the quantities x_i are not too large), and nearly equal to the "trading ratio" $-y_i/x_i = \gamma\bar{m}$ common to all of them. This provides a fairly strong intuitive rationale for expecting that a Cournot-Nash equilibrium will yield very nearly a Walras allocation in a large economy. In Section 6 we will verify this intuition.

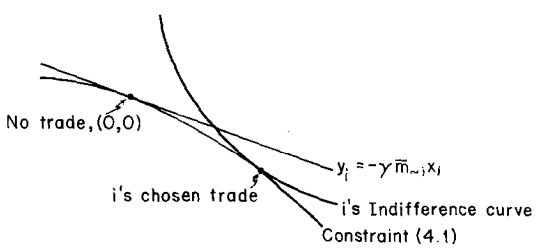

FIGURE 1

Intuition does not provide so firm a guide to the mechanism's disequilibrium performance. Certainly, at any play, in a large economy each player's response will be nearly the same as his Walras response, and therefore the amount that the price $\gamma\bar{m}$ adjusts will be nearly the Walrasian adjustment. But it is not so clear whether these similar adjustments at each step will provide convergence to the same limit. In Section 7 we verify that the Walras and Cournot adjustments do lead the economy to the same outcome.

5. Definitions

A *player* is characterized by his *utility function* $u: \mathbf{R}^2 \to \mathbf{R}$, which is always assumed to be increasing, strictly quasi-concave, and twice continuously differentiable. We will use $s(x, y)$ to denote the marginal rate of substitution, $s(x, y) = u_x/u_y$, and s_x and s_y will denote the partial derivatives of s.

We will only consider players for whom the Walras *net* demand function $D: \mathbf{R}_{++} \to \mathbf{R}$ is well-defined, where $D(p) = x$ if and only if $(x, -px)$ is the maximizer of u on the set $\{(\xi, \eta) \in \mathbf{R}^2 \mid p\xi + \eta \leqq 0\}$. Notice that

$$D'(p) = \frac{1 + xs_y}{s_x - ss_y}, \tag{5.1}$$

where the right-hand side of (5.1) is evaluated at $(x, y) = (D(p), -pD(p))$.

An *n-person economy* is an n-tuple $E = (u_1, \ldots, u_n)$ of players, and the *mean demand function* $\bar{D}$ of an n-person economy is the mean of the demand functions of its players:

$$\bar{D}(p) = \frac{1}{n} \sum_{i=1}^{n} D_i(p), \qquad \text{for each} \quad p \in \mathbf{R}_{++}.$$

A *Walras equilibrium* of an economy is a triple $(\hat{p}, \hat{x}, \hat{y}) \in \mathbf{R}_{++} \times \mathbf{R}^n \times \mathbf{R}^n$ that satisfies $\bar{D}(\hat{p}) = 0$, $\hat{x}_i = D_i(\hat{p})$ for all i, and $\hat{y}_i = -p\hat{x}_i$ for all i. A *Cournot equilibrium of the CW mechanism* with parameter value γ in E is an n-tuple $m^e = (m_1^e, \ldots, m_n^e) \in \mathbf{R}^n$ with the property

> For $i = 1, \ldots, n$: $\forall m_i \in \mathbf{R}$: $u_i(X_i(m^e), Y_i(m^e)) \geqq u_i(X_i(\tilde{m}), Y_i(\tilde{m}))$, where $\tilde{m} = (m_1^e, \ldots, m_{i-1}^e, m_i, m_{i+1}^e, \ldots, m_n^e)$, and $X_i(\cdot)$ and $Y_i(\cdot)$ are given by (1.1).

The properties of *large* economies will be analyzed via sequences of economies in which each term contains more players than the term before.

Formally, an *expanding sequence* of economies is a sequence $\{E(n)\}$ in which each term $E(n)$ contains n players. A *sequence of players* in $\{E(n)\}$, denoted $\{i(n)\}$, is a sequence in which the nth term is a player in the economy $E(n)$. An expanding sequence $\{E(n)\}$ is *bounded* if, at each pair $(x, y) \in \mathbf{R}^2$, the values s_i and $s_{ix} - s_i s_{iy}$ (the Gaussian curvature of u_i) are bounded and bounded away from zero over all sequences of players in $E(n)$. A sequence $\{m^e(n)\}$ is a *regular sequence of Cournot equilibria* of the CW mechanism in an expanding sequence $\{E(n)\}$ if there is a real sequence $\{\gamma(n)\}$ such that

for each n, $m^e(n)$ is a Cournot equilibrium of the CW mechanism with parameter-value $\gamma(n)$ in $E(n)$; (5.2)

$\{\gamma(n)\}$ is bounded and bounded away from zero; (5.3)

$\{X_i(m^e(n))\}$ and $\{Y_i(m^e(n))\}$ are bounded and bounded away from zero for every sequence $\{i(n)\}$ of players in $\{E(n)\}$; (5.4)

there is a number $\delta > 0$ such that $\{\bar{D}'(p(n); n)\}$ is bounded away from zero for every sequence $\{p(n)\}$ that satisfies $\gamma(n)\,\bar{m}^e(n) - \delta \leqq p(n) \leqq \gamma(n)\,\bar{m}^e(n) + \delta$. (5.5)

6. Cournot Equilibrium

Now we are prepared to verify that in a large economy every regular Cournot–Nash equilibrium will be very nearly the same as a Walras equilibrium. In a recent paper in this journal, Dubey, Mas Colell, and Shubik [2] established the *identity* of Cournot and Walras equilibria in continuum economies in a fairly broad class of mechanisms similar to the CW mechanism. Their result can be used to show the *near*-identity of the two equilibria in large but finite economies (i.e., in expanding sequences, but not in such general sequences as we are considering), and although their result does not apply to the CW mechanism, their proof could be adapted to do so. The proof that I will give here is, however, relatively brief and direct, and by avoiding the use of an infinite "limit" economy, it remains mathematically elementary (and remains applicable to sequences that do not have a continuum limit, so long as they are bounded in the sense of Section 5).

Theorem 1. *Let $\{E(n)\}$ be a bounded expanding sequence of economies, and let $m^e(n)$ be a regular sequence of Cournot equilibria of the CW mechanism in $\{E(n)\}$, with parameter values $\gamma(n)$. Then there is a sequence $\{\hat{p}(n); \hat{x}(n), \hat{y}(n)\}$ which satisfies*

(a) *for all sufficiently large* n, $(\hat{p}(n); \hat{x}(n), \hat{y}(n))$ *is a Walras equilibrium of* $E(n)$;

(b) $\lim_{n\to\infty} |\gamma(n)\, \bar{m}^e(n) - \hat{p}(n)| = 0$;

(c) *for every sequence* $\{i(n)\}$ *of players in* $\{E(n)\}$,

$$\lim_{n\to\infty} |\hat{x}_i - X_i(m^e(n))| = \lim_{n\to\infty} |\hat{y}_i - Y_i(m^e(n))| = 0.$$

Proof. The following notation will be helpful:

$$x_i^e = X_i(m^e(n)) \qquad \text{and} \qquad y_i^e = Y_i(m^e(n));$$

$$\tilde{x}_i = D_i(\gamma(n)\, \bar{m}^e(n); n) \qquad \text{and} \qquad \tilde{y}_i = -\gamma(n)\, \bar{m}^e(n)\, \tilde{x}_i.$$

The proof will consist of the following three lemmas:

LEMMA 1. $\tilde{x}_{i(n)} - x_{i(n)}^e \to 0$ *and* $\tilde{y}_{i(n)} - y_{i(n)}^e \to 0$ *for every sequence* $\{i(n)\}$ *of players.*

LEMMA 2. *There is a sequence* $\{\hat{p}(n)\}$ *for which* $\hat{p}(n) - \gamma \bar{m}^e(n) \to 0$ *and for which* $\bar{D}(\hat{p}(n); n)$ *is eventually zero.*

LEMMA 3. *If* $\{\hat{p}(n)\}$ *is as in Lemma* 2, *then for every sequence* $\{i(n)\}$ *of players,* $\hat{x}_{i(n)} - x_{i(n)}^e \to 0$ *and* $\hat{y}_{i(n)} - y_{i(n)}^e \to 0$, *where* $\hat{x}_i = D_i(\hat{p}(n))$ *and* $\hat{y}_i = -\hat{p}(n)\, D_i(\hat{p}(n))$.

Proof of Lemma 1 (see Fig. 2). No ambiguity will result if we shorten "$i(n)$" to "i." Assume that $x_i^e > 0$, from which it follows that $\tilde{x}_i > x_i^e$ (for $x_i^e < 0$ the proof is identical, except that $\tilde{x}_i < x_i^e$; for $x_i^e = 0$, we have $\tilde{x}_i = x_i^e$). For each n, define $Q(\cdot; n)$, $I(\cdot; n)$, and $L(\cdot; n)$ as follows:

$$Q(x; n) = -\gamma(n)\, \bar{m}_{\sim i}^e(n) x - \frac{1}{n-1} \gamma(n)\, x^2,$$

$$I(x; n) \text{ is the } y \text{ for which } u_i(x, y) = u_i(x_i^e, y_i^e),$$

$$L(x; n) = y_i^e - \gamma(n)\, \bar{m}^e(n)(x - x_i^e).$$

That is, Q is the quadratic constraint (4.1), I is the indifference curve through (x_i^e, y_i^e), and L is the "price-taking" constraint, as depicted in Fig. 2. Notice that for $x_i^e < x < x_i$, I lies between Q and L: $Q(x; n) < I(x; n) < L(x; n)$.

Now suppose that $\tilde{x}_i - x_i^e$ does not vanish as $n \to \infty$; without loss of generality, suppose that there is a $\delta > 0$ such that $\tilde{x}_i > x_i^e + \delta$ for all n. Since $L(x(n); n) - Q(x(n); n)$ vanishes for all sequences $\{x(n)\}$ lying between $\{x_i^e\}$

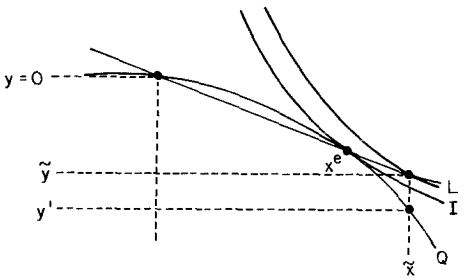

FIGURE 2

and $\{\tilde{x}_i\}$, we have (by a double application of the Mean Value Theorem to each function $I(\cdot; n)$) a sequence $x(n)$ for which $I''(x(n); n) \to 0$—i.e., for which the Gaussian curvature of u_i vanishes, which violates the boundedness of the expanding sequence $\{E(n)\}$, thereby completing the proof of Lemma 1.

Proof of Lemma 2. Suppose that there is not such a sequence; then (because $\{m^e(n)\}$ is a regular sequence of Cournot equilibria) there exist both a $\delta > 0$ and a subsequence along which each $\bar{D}(\cdot; n)$ is (without loss of generality: the other cases are treated identically) positive and decreasing for all p in the interval

$$\gamma(n)\, \bar{m}^e(n) - \delta \leqq p \leqq \gamma(n)\, \bar{m}^e(n) + \delta,$$

and along which, for any sequence $\{p(n)\}$ satisfying

$$\gamma(n)\, \bar{m}^e(n) - \delta \leqq p(n) \leqq \gamma(n)\, \bar{m}^e(n) + \delta \text{ for all } n$$

we have $\bar{D}'(p(n); n)$ bounded away from zero. Thus, for each n (along the subsequence), we have

$$0 < \bar{D}(\gamma(n)\, \bar{m}^e(n) + \delta; n) < \bar{D}(\gamma(n)\, \bar{m}^e(n); n);$$

moreover, Lemma 1 guarantees that $\bar{D}(\gamma(n)\, \bar{m}^e(n); n) \to 0$. Therefore, if we apply the mean value theorem to each function $\bar{D}(\cdot; n)$, between $\gamma(n)\, \bar{m}^e(n)$ and $\gamma(n)\, \bar{m}^e(n) + \delta$, we obtain a sequence $\{p(n)\}$ satisfying both $\bar{D}'(p(n); n) \to 0$ and $\gamma(n)\, \bar{m}^e(n) \leqq p(n) \leqq \gamma(n)\, \bar{m}^e(n) + \delta$, which contradicts the fact that $\bar{D}'(p(n); n)$ is bounded away from zero, thereby completing the proof of Lemma 2.

Proof of Lemma 3. Let $\{i(n)\}$ be an arbitrary sequence of players in $\{E(n)\}$; we must show that $x^e_{i(n)} - \hat{x}_{i(n)}$ vanishes as $n \to \infty$, and it will follow from boundedness of $\gamma(n)\, \bar{m}^e(n)$ and Lemma 2 that $y^e_{i(n)} - \hat{y}_{i(n)}$ also vanishes.

Furthermore, because of Lemma 1 it will suffice to show that $\tilde{x}_{i(n)} - \hat{x}_{i(n)}$ vanishes—i.e., that

$$\lim_{n \to \infty} |D_i(\gamma(n)\, \bar{m}^e(n); n) - D_i(\hat{p}(n); n)| = 0.$$

If this limit were not zero, Lemma 2 would enable us, by applying the mean value theorem to each term, to obtain a sequence $\{p(n)\}$ in which $D_i'(p(n); n)$ is unbounded, which violates the boundedness of $\{E(n)\}$, and this contradiction completes the proof. ∎

7. Stability under Cournot Behavior

Before we begin our analysis of the stability of the CW mechanism under Counot behavior, it will be helpful to recall the conventional Walrasian tatonnement analysis, in which, at discrete intervals of time, the price is adjusted by an amount proportional to the mean excess demand that has just been reported:

$$p(t+1) = p(t) + k\bar{D}(p(t)). \tag{7.1}$$

The parameter k is the "speed" at which price is being adjusted. If we combine (7.1) with the linear Taylor approximation to $\bar{D}(p(t))$, near a Walras equilibrium $\hat{p}$, namely,

$$\bar{D}(p(t)) \approx \bar{D}(\hat{p}) + \bar{D}'(\hat{p})(p(t) - \hat{p}),$$

then we obtain

$$\begin{aligned} p(t+1) - \hat{p} &\approx p(t) - \hat{p} + k\bar{D}(\hat{p}) + k\bar{D}'(\hat{p})(p(t) - \hat{p}) \\ &= [1 + k\bar{D}'(\hat{p})](p(t) - \hat{p}). \end{aligned}$$

Thus, an interior equilibrium $\hat{p}$ will be locally stable if $-2 < k\bar{D}'(\hat{p}) < 0$—i.e., if the mean demand function is downward-sloping at $\hat{p}$, but is not too elastic for the adjustment speed k. Thus, if it is known that the demand elasticity is less than some upper bound, then the tatonnement can be "stabilized" by ensuring that k is sufficiently small—specifically, that $k\,|D'(p)| < 2$. In this section we will establish an analogous result for the CW mechanism under Cournot best-replay behavior: if the economy is large enough, then any regular Cournot equilibrium m^e will be locally stable, so long as the CW parameter γ satisfies $\gamma\,|\bar{D}'(\gamma\bar{m}^e)| < 2$.

The analysis will be carried out in terms of the *transition function* $F = (F_1, ..., F_n)\colon \mathbf{R}^n \to \mathbf{R}^n$ defined by $m(t+1) = F(m(t))$ under Cournot best-replay behavior. Thus, each $F_i\colon \mathbf{R}^n \to \mathbf{R}$ is a player's *reaction function*, giving

his best message $m_i(t+1)$ against each possible n-tuple $m(t)$. A Cournot equilibrium is an n-tuple m^e that satisfies $F(m^e)=m^e$, and we will show that Cournot equilibria are stable (in large economies) by establishing that all n roots of the Jacobian matrix of F lie within the unit circle of the complex plane.

Let us write A for the Jacobian matrix of F, an $n \times n$ matrix with entries

$$a_{ij}=\begin{cases} 0, & \text{if } i=j \\ \dfrac{\partial m_i(t+1)}{\partial m_j(t)}, & \text{if } i \neq j. \end{cases}$$

For $i \neq j$ we can obtain an expression for a_{ij} by applying the implicit function theorem to the first-order condition of player i's maximization problem. A little manipulation of Eq. (4.2) yields the following convenient form of that first-order condition:

$$(n-1)\, s_i-(n-2)\,\gamma\bar{m}-\gamma m_i=0. \tag{7.2}$$

Writing $G^i(m)$ for the left-hand side of (7.2), we have

$$a_{ij}=-\frac{\partial G^i/\partial m_j}{\partial G^i/\partial m_i}, \qquad \text{if } i \neq j, \tag{7.3}$$

$$\begin{aligned}\frac{\partial G^i}{\partial m_j}&=-\frac{n-2}{2}\gamma+(n-1)\frac{\partial y_i}{\partial m_j}s_{iy}+(n-1)\frac{\partial x_i}{\partial m_j}s_{ix}\\ &=-\frac{n-2}{n}\gamma+(n-1)\left(-\frac{1}{n}\gamma x_i+\frac{1}{n}\gamma\bar{m}\right)s_{iy}-(n-1)\frac{1}{n}s_{ix}\\ &=-\frac{n-2}{n}\gamma-\frac{n-1}{n}\gamma x_i s_{iy}+\frac{n-1}{n}(\gamma\bar{m}s_{iy}-s_{ix})\end{aligned} \tag{7.4}$$

and

$$\begin{aligned}\frac{\partial G^i}{\partial m_i}&=-\frac{n-2}{n}\gamma-\gamma+(n-1)\frac{\partial y_i}{\partial m_i}s_{iy}+(n-1)\frac{\partial x_i}{\partial m_i}s_{ix}\\ &=-\frac{n-2}{n}\gamma-\gamma+(n-1)\left(-\frac{1}{n}\gamma x_i-\gamma\bar{m}\left(1-\frac{1}{n}\right)\right)s_{iy}\\ &\quad+(n-1)\left(1-\frac{1}{n}\right)s_{ix}\\ &=-\frac{n-2}{n}\gamma-\gamma-\frac{n-1}{n}\gamma x_i s_{iy}+\frac{n-1}{n}(n-1)(s_{ix}-\gamma\bar{m}s_{iy}).\end{aligned} \tag{7.5}$$

Notice that, for each i, $\partial G_i/\partial m_j$ (and therefore a_{ij}) is independent of j.

THEOREM 2. *Let $\{E(n)\}$ be a bounded expanding sequence of economies, and let $\{m^e(n)\}$ be a regular sequence of Cournot equilibria of the* CW *mechanism in $\{E(n)\}$ under the sequence $\{\gamma(n)\}$ of parameter values. If $-2 < \gamma(n)\,\bar{D}'(\gamma\bar{m}^e(n); n) < 0$ for all n, then the equilibria $m^e(n)$ are eventually locally stable under Cournot best-replay behavior.*

Proof. We will show that eventually all roots of the Jacobian matrices $A(n)$ have modulus less than unity. We have just shown, above, that the off-diagonal elements of each row of a matrix $A(n)$ are identical to one another—i.e., $a_{ij}(n)$ is independent of j for $j \neq i$. It will be helpful to denote this common value by $\alpha_i(n)$:

$$a_{ij}(n) = \begin{cases} 0, & \text{if } j = i \\ \alpha_i(n), & \text{if } j \neq i. \end{cases}$$

Notice from (7.4) and (7.5) that for any sequence $\{\alpha_i(n)\}$, the sequence $\{n\alpha_i(n)\}$ is bounded.

For each n, let $r(n)$ be a dominant root of $A(n)$, i.e., a root whose modulus is not exceeded by that of any other root of $A(n)$. We must show that eventually $|r(n)| < 1$. For each n, let $x(n) = (x_1(n),..., x_n(n))$ be an eigenvector of $A(n)$ corresponding to the root $r(n)$. For each n and each i $(i = 1,..., n)$, we have

$$\begin{aligned} r(n)\,x_i(n) &= \sum_{j=1}^{n} a_{ij}(n)\,x_j(n) \\ &= n\alpha_i(n)\frac{1}{n}\sum_{j=1}^{n} x_j(n) - \alpha_i(n)\,x_i(n) \\ &= n\alpha_i(n)\,\bar{x}(n) - \alpha_i(n)\,x_i(n). \end{aligned} \tag{7.6}$$

Summing over $i = 1,..., n$ and dividing by n, we obtain

$$r(n)\,\bar{x}(n) = \bar{x}(n)\sum_{i=1}^{n} \alpha_i(n) - \frac{1}{n}\sum_{i=1}^{n} \alpha_i(n)\,x_i(n). \tag{7.7}$$

Without loss of generality, let the sequence $x_1(n)$ of first components of the eigenvectors $x(n)$ satisfy both

$$|x_1(n)| \geqq |x_i(n)|, \qquad i = 2,..., n \tag{7.8}$$

and

$$\{x_1(n)\} \text{ is bounded and bounded away from zero.} \tag{7.9}$$

Since $x_1(n) \neq 0$ for each n, it follows from (7.6) that

$$r(n) = n\alpha_1(n)\frac{\bar{x}(n)}{x_1(n)} - \alpha_1(n) \qquad \text{for each } n. \tag{7.10}$$

Now consider any subsequence of $\{r(n)\}$ which is bounded away from zero; if we can show that any such subsequence eventually satisfies $|r(n)| < 1$, then the proof will be complete (as it would be, too, of course, if there *is* no such subsequence of $\{r(n)\}$). Since $\{n\alpha_1(n)\}$ is bounded, as we observed at the beginning of the proof, and since $\{x_1(n)\}$ is bounded away from zero, it follows from (7.10) that the corresponding subsequence of $\{\bar{x}(n)\}$ must be bounded away from zero, just as the subsequence of $\{r(n)\}$ is. Thus, we can rewrite (7.7) as

$$r(n) - \sum_{i=1}^{n} \alpha_i(n) = -\frac{1}{n} \sum_{i=1}^{n} \frac{x_i(n)}{\bar{x}(n)} \alpha_i(n) \tag{7.11}$$

for this subsequence, and since $\{\bar{x}(n)\}$ is bounded away from zero (and $\{x_i(n)\}$ is bounded and $\alpha_i(n) \to 0$), the right-hand side of (7.11) vanishes along the subsequence. Consequently, the distance between $r(n)$ and $\sum_1^n \alpha_i(n)$ vanishes, and the proof is therefore completed by the following demonstration that $|\sum_1^n \alpha_i(n)|$ is eventually less than unity:

$$\lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \alpha_i(n) - [1 + \gamma(n)\, \bar{D}'(\gamma(n)\, \bar{m}^e(n); n)] \right\}$$

$$= \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \frac{\dfrac{n-2}{n}\gamma(n) + \dfrac{n-1}{n}\gamma(n)\, x_i s_{iy} - \dfrac{n-1}{n}(\gamma(n)\, \bar{m}^e(n)\, s_{ij} - s_{ix})}{(n-1)\dfrac{n-1}{n}(s_{ix} - \gamma(n)\, \bar{m}^e(n)\, s_{iy}) - \dfrac{n-1}{n}\gamma(n)\, s_{iy} - 2\dfrac{n-1}{n}\gamma(n)} \right.$$

$$\left. - [1 + \gamma(n)\, \bar{D}'(\gamma(n)\, \bar{m}^e(n); n] \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{\gamma(n) + \gamma(n)\, x_i s_{iy} - (\gamma(n)\, \bar{m}^e(n)\, s_{iy} - s_{ix})}{\left(\dfrac{n-1}{n}\right)^2 (s_{ix} - \gamma(n)\, \bar{m}^e(n)\, s_{iy}) - \dfrac{n-1}{n^2}\gamma(n)(x_i s_{iy} + 2)} \right.$$

$$\left. - [1 + \gamma(n)\, \bar{D}'(\gamma(n)\, \bar{m}^e(n); n)] \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{(n)(1 + x_i s_{iy}) + (s_{ix} - \gamma(n)\, \bar{m}^e(n)\, s_{iy})}{s_{ix} - \gamma(n)\, \bar{m}^e(n)\, s_{iy}} \right.$$

$$\left. - [1 + \gamma(n)\, \bar{D}'(\gamma(n)\, \bar{m}^e(n); n)] \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[1 + \gamma(n) \frac{1 + x_i s_{iy}}{s_{ix} - \gamma(n)\, \bar{m}^e(n)\, s_{iy}} \right] \right.$$

$$\left. - [1 + \gamma(n)\, \bar{D}'(\gamma(n)\, \bar{m}^e(n); n)] \right\}$$

$$= \lim_{n\to\infty} \left\{ \gamma(n) \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1 + x_i s_{iy}}{s_{ix} - \gamma(n)\, \bar{m}^e(n)\, s_{iy}} - D_i'(\gamma(n)\, \bar{m}^e(n); n) \right] \right\},$$

which is zero, because wherever an individual's demand function is differentiable is satisfies

$$D_i'(p) = \frac{1 + x_i s_{iy}}{s_{ix} - p s_{iy}}.$$

Since $|1 + \gamma(n)\, \bar{D}|(\gamma(n)\, \bar{m}^e(n); n)| < 1$, we have $|\sum_{i=1}^{n} \alpha_i(n)| < 1$ for sufficiently lare n. ∎

8. Concluding Remarks

The Cournot–Walras mechanism that I have introduced here is the simplest of all the Cournot-inspired general equilibrium mechanisms that have recently begun to appear, and it establishes that it is possible to devise "auctioneerless" resource allocation methods that (in large economies) combine several desirable features: the achievement of virtually Walrasian outcomes; the duplication of the stability (or "stabilizibility") of the Walrasian tatonnement; and the use of information transfers of no larger dimension than the number of non-numeraire goods.

As I pointed out in the Introduction, the economic setting in which the results have been established in Sections 6 and 7 is a simple and rather special one in several respects. In all but one of these respects, it is clear how to generalize the setting and how (at the cost of greater length, more cumbersome notation, etc.) to obtain the same results. The use of best-replay behavior in the analysis is, however, not so innocuous. It would clearly be better to have a real theory of disequilibrium behavior. Lacking such a theory, however, best-replay behavior is a natural starting point. There is even a bit of experimental evidence (see [11, 23]) suggesting that people may tend to practice approximately best-replay behavior in such settings, so long as they perceive that it is resulting in a rapid convergence to an equilibrium. And I have already pointed out in the Introduction the substantial theoretical use of best-replay behavior in lieu of a more adequate theory.

In any case, the enormous disparity under best-replay behavior between, on the one hand, the Walras-like stability demonstrated here and, on the

other hand, the severe instability of similar mechanisms for making public-goods decisions suggests the sort of striking and important results that we might hope to obtain when we have a more fully-developed theory of disequilibrium behavior.

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