

## Notes, Comments, and Letters to the Editor

### On the Existence of Maximal Elements

T. Bergstrom has recently shown [1] that any acyclic binary relation which is defined on a compact topological space in such a way that all of its lower-contour sets are open, must yield a maximal element of the space. This result has been discovered independently by several people; see, for example, [2, 4, 6], as well as [1, footnote 1]. The result is in fact implicit in [5], as will be shown below. Bergstrom's proof is rather indirect (it proceeds via the relation's transitive closure), and it makes use of a form of the axiom of choice. A straightforward and elementary proof is provided here, and the possibility of generalizing the result is then considered.

Let us first define our terms. For a binary relation  $R$  on a set  $X$ , and a positive integer  $n$ , an  $(R, n)$ -cycle (sometimes just "an  $R$ -cycle," or "an  $n$ -cycle") is an  $n$ -tuple  $(x_1, \dots, x_n)$  of members of  $X$  which satisfies  $(x_n, x_1) \in R$  and, for  $1 \leq i \leq n - 1$ ,  $(x_i, x_{i+1}) \in R$ .  $R$  is said to be acyclic if it has no cycles. A member  $x$  of  $X$  is said to  $R$ -dominate a member  $y$  if  $(x, y) \in R$ ; an  $R$ -maximal member of a subset  $A$  of  $X$  is one which is not dominated by any member of  $A$ . An  $n$ -tuple (or set) of members of  $X$  is  $R$ -dominated if each of its components (or members) is. Let  $xR$  and  $Rx$  denote the lower- and upper-contour sets of a member  $x$  of  $X$ :  $xR = \{y \in X \mid (x, y) \in R\}$  and  $Rx = \{y \in X \mid (y, x) \in R\}$ . Notice that a subset  $A$  of  $X$  has an  $R$ -maximal member if and only if the collection  $\{xR \mid x \in A\}$  does not cover  $A$ .

Now let  $X$  be a compact topological space, and let  $P$  be an acyclic relation on  $X$ , all of whose lower-contour sets are open. Suppose that  $X$  has no  $P$ -maximal element; then  $\{xP \mid x \in X\}$  is an open cover of  $X$ . Since  $X$  is compact there is a finite subset  $A$  of  $X$  for which  $\{xP \mid x \in A\}$  covers  $X$  and a fortiori covers  $A$ ; hence,  $A$  has no  $P$ -maximal member. It is easy to verify, however, that if  $P$  is acyclic, then every finite subset of  $X$  must have a  $P$ -maximal member, and this contradiction establishes that  $X$  must indeed have a  $P$ -maximal element.

An important virtue of this proof is the way in which it lays bare the roles of acyclicity, compactness, and open lower-contour sets: the latter two conditions (which we will call "the topological conditions") together guarantee that  $X$  has a maximal element if each of its finite subsets does, and acyclicity is equivalent to the existence of a maximal member of each finite subset.

Now in order that  $X$  include a  $P$ -maximal element, it is clearly not necessary that every finite subset do so as well; in other words, we should be able to weaken the requirement that  $P$  be acyclic. Sonnenschein [5] has shown that if  $X$  is a convex subset of a Euclidean space, then we can indeed substitute asymmetry<sup>1</sup> of  $P$  for acyclicity, if we also require that  $P$  be (upper-)convex. The remarkable result recently obtained by Shafer and Sonnenschein [3] contains as a special case the further weakening (when  $P$  is convex) of asymmetry to irreflexivity (no 1-cycles).

Sonnenschein's proof in [5] consists of an elegant demonstration that each finite subset of  $X$  is weakly dominated, as expressed in the following condition.

*Condition 1.* For every finite subset  $A$  of  $X$ , there is an element  $x \in X$  such that  $(y, x) \notin P$  for every  $y \in A$ .

Condition 1 is clearly implied by acyclicity of  $P$ , but is in general weaker than acyclicity. In the presence of the two topological conditions, however, Condition 1 implies (and is therefore equivalent to) acyclicity: the set  $A$  which was constructed in the above proof violates Condition 1. Although Sonnenschein did not state this result explicitly in [5], both a statement and a proof of it (essentially the proof described above) are implicit in the first sentence of his proof of Theorem 4 (his reference there to "Theorem 2" should read "Theorem 1.")

The following condition is even weaker still: It is implied by, but does not generally imply, Condition 1.

*Condition 2.* For every finite subset  $A$  of  $X$  which includes a  $P$ -cycle, there is an element  $x \in X$  such that  $(y, x) \notin P$  for every  $y \in A$ .

Just as before, if we add the two topological conditions, then Condition 2 implies (and is therefore equivalent to) the existence of a  $P$ -maximal element of  $X$ . In order to see this, notice that the set  $A$  which was constructed in the proof includes a cycle, and that it therefore violates Condition 2 as well as Condition 1.

One is led inexorably to Condition 3, below, which is even weaker still. Indeed, Sonnenschein's own description of his proof (in [5, Example 2]) substitutes Condition 3 for Condition 1 (for an asymmetric relation  $P$ ).

*Condition 3.* For each  $P$ -cycle  $C$ , there is an element  $x \in X$  such that  $(y, x) \notin P$  for each component  $y$  of  $C$ .

When the set  $X$  is finite, of course, Conditions 2 and 3 are equivalent. When  $X$  is infinite, however, as the following counterexample demonstrates,

<sup>1</sup>  $P$  is asymmetric if it has no 2-cycles (and therefore no 1-cycles, either);  $P$  is convex if each of its upper-contour sets  $Px$  is convex.

Condition 3 (even for an asymmetric relation  $P$ ) is not sufficient, in the presence of the two topological conditions, to guarantee the existence of a  $P$ -maximal element of  $X$ .

COUNTEREXAMPLE. Let  $X$  be the set of nonnegative integers, endowed with the topology in which the only sets which are not closed are the infinite sets which do not contain 0 (in other words, 0 is the only accumulation point in  $X$ , and it is an accumulation point of every infinite subset).<sup>2</sup> Notice that  $X$  is compact. Let  $P$  be defined as follows:

$$\begin{array}{ll} 0P = \{3\}, & 3P = \emptyset, \\ 1P = \{0; 4, 5, 6, \dots\}, & 4P = \{2\}, \\ 2P = \{1\}, & nP = \emptyset, \text{ if } n > 4. \end{array}$$

It is easy to verify that  $P$  is asymmetric, satisfies Condition 3, and has open lower-contour sets, but  $X$  clearly has no  $P$ -maximal element.

Finally, it should be noted that some results of the kind that we have been considering are more useful than others. We are generally interested not in a maximal element of just a single set  $X$ , but rather in a whole family  $\mathcal{F}$  of subsets of some underlying set  $X$ , and in whether *each* member of  $\mathcal{F}$  has a maximal element. For example,  $\mathcal{F}$  might be the collection of all possible budget sets (in the theory of the consumer); the family of all possible production sets (in the theory of production); or, more generally, the family of all feasible sets which could obtain (in a programming problem of any kind). In order for any set of conditions on a relation  $P$  to be useful in this context, then, the conditions should be "inherited" when one passes from the space  $X$  on which  $P$  is defined, to any member of some "interesting" family of subsets of  $X$ . Both acyclicity and the topological condition on  $P$  are, of course, inherited when one passes to *any* subset of the underlying space (if we use each subset's relative topology); therefore, the following, more interesting form of our result is valid.

**THEOREM.** *Let  $X$  be a topological space, and let  $P$  be a binary relation on  $X$ , all of whose lower-contour sets  $xP$  are open. If  $P$  is acyclic, then every compact subset of  $X$  has a  $P$ -maximal member.*

Asymmetry and irreflexivity of a relation  $P$ , like acyclicity, are inherited when passing to any subset of the underlying set  $X$  on which  $P$  is defined. When  $X$  is a subset of a linear space, (upper-)convexity of  $P$  is inherited when

<sup>2</sup> A subset of  $X$  is open (i.e. a member of the topology) if it is cofinite (has a finite complement) or does not contain 0. The function  $h: X \rightarrow \mathbb{R}$  defined by  $h(0) = 0$  and  $h(n) = 1/n$  is a homeomorphism of  $X$  with the subset  $\{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$  of  $\mathbb{R}$ , with its usual topology.

passing to convex subsets of  $X$ . Hence, the Euclidean results mentioned above can be expressed in the same, more useful form as the theorem we have just given: Let  $X$  be a subset of a Euclidean space, and let  $P$  be a binary relation on  $X$ , all of whose lower-contour sets are open; if  $P$  is irreflexive and convex, then every compact convex subset of  $X$  has a  $P$ -maximal element<sup>3</sup>.

The distinction between conditions which are inherited and ones which are not, and between the two kinds of results that one can obtain, is not a vacuous one, and is in fact rather important. We have already seen that Conditions 1 and 2 can each be substituted for acyclicity, without affecting the validity of the result with which this note began; however, neither condition is inherited<sup>3</sup> on any apparently useful family of subsets, and the theorem we have just stated is *not* valid when either condition is substituted for acyclicity. Consider, for example, the relation  $P = \{(a, a)\}$  on the discrete space  $X = \{a, b\}$ ; Conditions 1 and 2 are both satisfied on  $X$ , but not on the subset  $\{a\}$ , and that subset, although compact, does not have a maximal member. An irreflexive example is provided by  $P = \{(a, b), (b, a)\}$  on  $X = \{a, b, c\}$ .

It is an open question whether there exist conditions weaker than acyclicity which are sufficient (with the topological conditions) to insure a maximal element, and which are still inherited when passing to compact subsets. Peter Fishburn, however, has pointed out to me that there is no such condition which is also *necessary* for existence of a maximal element. For suppose that there *were* such a condition, which we will call Condition C; then we can show that the existence of a  $P$ -maximal element is *itself* a condition which is inherited on compact sets (when  $P$  has open lower-contour sets). Let  $X$  be a topological space, and let  $P$  be a relation on  $X$  which has open lower-contour sets, and which has a maximal element in  $X$ . Then  $P$  satisfies Condition C (because C is necessary); hence, in every compact subset of  $X$ , the restriction of  $P$  also satisfies Condition C (because C is inherited); and finally, each compact subset has a  $P$ -maximal member (because C is sufficient, and the topological condition on  $P$  is inherited). However, we have just seen in the preceding paragraph how easy it is to construct relations which have open lower-contour sets, but which do not yield maximal elements on each compact subset.

#### ACKNOWLEDGMENTS

The author is indebted to Peter Fishburn, who suggested Conditions 1 and 2 and the observation contained in the final paragraph. Reference [4] is due to Robert Wilson.

<sup>3</sup> In his proof in [5], Sonnenschein actually demonstrates that convexity of  $P$  yields a stronger form of Condition 1: that each finite subset must not only be weakly dominated, but also that at least one of the dominating (i.e., undominated) elements must lie in the set's convex hull. The condition in this form is inherited when passing to convex subsets.

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RECEIVED: December 2, 1975; REVISED: March 15, 1976

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