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On the Informational Size of Message Spaces*

In a recent paper, Mount and Reiter established that, in a certain sense, the competitive mechanism is an "informationally most efficient" procedure for allocating resources. This result, of course, depends upon the way in which we characterize the notion of informational efficiency. Several alternative characterizations, and the relationships among them, are given here, and it is shown under which characterizations the above result is true, and under which it is false. It is shown that there is an intuitively appealing "best" characterization for which it is true.

In a recent paper,[1] which I will denote by [MR], Mount and Reiter have introduced both the notion of an "informationally admissible" or "feasible" procedure for allocating resources, and a formal characterization of the relative sizes of the topological spaces that are used to convey information in such procedures. Then, with these two concepts providing the formal groundwork, Mount and Reiter have gone on to establish the "informational efficiency" that has been held to be an important virtue of the competitive mechanism. Specifically, they show that among all procedures which are capable of achieving competitive (and therefore optimal) allocations, the competitive procedure itself has the smallest space of equilibrium messages. In other words, the competitive procedure is, in a certain sense, the most efficient procedure that one could devise.

This "optimality" of the competitive procedure was demonstrated in [MR] in both a global context and a local context. As we will see in a moment, however, the global result is untenable. The informal result—that the competitive procedure is somehow the most efficient one from an informational point of view—nevertheless remains very appealing, as does the approach that Mount and Reiter have taken to deal with the question. This suggests that we ought to reconsider the formal structure of one or both of the concepts introduced in [MR]. Accordingly, several alternative characterizations of the relative informational size of spaces will be given, and it will be shown that the global efficiency theorem is recovered when we use some of these characterizations (and that it holds over a broader class of procedures than in [MR]). Furthermore, it will be shown that "Frechet size" (probably the most

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[1] K. Mount and S. Reiter, The informational size of message spaces, *J. Econ. Theory* **8** (1974), 161–192.

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intuitively appealing concept of size) is the unique strongest characterization for which the efficiency theorem holds. The results that will be established here lend support, then, to the notion of admissibility developed by Mount and Reiter, on the one hand, and, on the other hand, to the concept of Frechet size as probably the most useful one for comparing the informational capacity of topological spaces.

I will use the notation and definitions of [MR], where n and l are the numbers of agents and of commodities, respectively (and $n \geqslant 2$), and where S denotes the "open" unit simplex in $\mathbf{R}^l$. I will denote procedures generically by (μ, g), and their message spaces by M; the competitive procedure, as defined in [MR], will be denoted by $(\bar{\mu}, \bar{g})$, and its message space by $\bar{M}$. $\bar{M}$ is homeomorphic to $S \times \mathbf{R}^{(n-1)(l-1)}$.[2]

The main results in [MR] concern the size of the message space of any procedure which could be used to attain the same allocations as the competitive procedure. The notion that one topological space is larger, or can convey more information, than another, is given formal expression by Mount and Reiter in the following definition (some examples, which will help to illuminate the definition, will be provided shortly).[3]

DEFINITION 1.[4] A space X has as much information as a space Y, which we denote by $X \cdot\!\geqslant Y$, if there is a continuous, locally sectioned surjection from X to Y.

Theorem 31 of [MR] (together with an extension, Corollary 34) states that if (μ, g) is an admissible procedure which uses a message space M to realize the competitive outcome function on classical exchange economies, and if the message correspondence μ is upper semicontinuous (u.s.c.), then $M \cdot\!\geqslant \bar{M}$. Theorem 35 establishes, without the requirement that μ be u.s.c., that M is *locally* larger than $\bar{M}$, so long as M is Hausdorff.

Theorem 31 (and a fortiori Corollary 34) is the result that, as I mentioned before, is incorrect.[5] Let us take a closer look, then, at the concept of the relative size of spaces, as characterized in Definition 1, in order to see whether

[2] In [MR], the notation $S \times Y$ is used instead of $\bar{M}$. This is confusing, partly because they use Y for two distinct (nonhomeomorphic) spaces.

[3] See also pp. 173–180 of [MR], and "On the Definition of Informational Size," by Mount and Reiter (D.P. #140 of The Center for Mathematical Studies in Economics and Management Science, December 1974).

[4] "Space" will always mean "topological space." Unless otherwise noted, a subset will always have the relative topology, the one it inherits as a subspace, and a product will always have the product topology.

[5] The example on p. 189 of [MR] is intended to show that the theorem would fail if μ were not required to be u.s.c.; however, the correspondence in the example *is* u.s.c., and the example is thus a counterexample to the truth of the theorem. The misstep in the proof given in [MR] consists in inferring that a correspondence which has a closed graph is u.s.c.

we can find a reasonable alternative characterization for which the theorem might be true.[6]

First of all, it is easy to verify that the relation $\cdot\!\geqslant$ is both a quasi-ordering (reflexive and transitive) and a topological invariant, just as we would expect a notion of "size" to be. Consideration of a few simple examples will tell us somewhat more about the ordering $\cdot\!\geqslant$. Let $\mathbf{R}$ be the set of real numbers; let I, $\dot{I}$, and $\mathring{I}$ be the intervals $[0, 1]$, $[0, 1)$, and $(0, 1)$, respectively; let I^n be the n-dimensional cube (n-fold product of I); let C be the circle in $\mathbf{R}^2$; and let Q be the set of all rational numbers. Let all of those sets be endowed with their usual topologies, and let D be any finite (or indeed any countable) discrete space of more than one member. Then we have,[7]

$$\mathbf{R} \doteq \mathring{I} \cdot\!> \dot{I} \cdot\!> I \doteq C,$$

$$I^n \cdot\!> I \doteq C, \qquad \text{and} \qquad Q \cdot\!> D.$$

However, none of the remaining pairs is related by $\cdot\!\geqslant$; in particular, $I^n \cdot\!\not\geqslant \mathbf{R}$, $\mathbf{R} \cdot\!\not\geqslant Q$, and $\mathbf{R} \cdot\!\not\geqslant D$.

Now, if $\geqslant$ denotes a quasi-ordering which relates topological spaces in terms of their "size," then, on the face of it, it certainly seems reasonable to expect that $\mathbf{R} > D$, and even that $I^n > \mathbf{R}$ and $\mathbf{R} > Q$, or at least $\mathbf{R} \geqslant Q$. Going a bit further in the same vein, we might even be led to expect that whenever one space is a subspace of another, say $S \subseteqq X$, then the spaces will stand in the relation $S \leqslant X$ (this would yield $\mathbf{R} \doteq \mathring{I} \doteq \dot{I} \doteq I$, for example.) This principle, together with the ideas underlying Definition 1, leads us to the following definition.

DEFINITION 2. $X \geqslant_S Y$ if and only if there is a subspace X' of X for which $X' \cdot\!\geqslant Y$.

The relation $\geqslant_S$ is still a quasi-ordering and a topological invariant; it is the smallest quasi-ordering which both includes $\cdot\!\geqslant$ and satisfies the condition that $X \subseteqq Y$ implies $X \leqslant Y$. We could go on indefinitely, of course, specifying new quasi-orderings. The four that are given in the following definition are probably the most intuitive ones, however, and are therefore useful; moreover, one of the four will turn out to be just the ordering we are looking for.

[6] Another possible approach to recovering the theorem is to replace the condition "μ is u.s.c." with some other reasonable restriction on procedures (and to retain the ordering $\cdot\!\geqslant$). It has been suggested, for example, that the theorem is true if we substitute the condition "μ^{-1} is u.s.c." This condition cannot be called a "reasonable" one, however, since few, if any, procedures satisfy it; the competitive procedure, in particular, does not. The force of Theorem 2 below is to show that *no* such condition will recover Theorem 31 with the ordering $\cdot\!\geqslant$.

[7] As usual, $X \cdot\!> Y$ means "$X \cdot\!\geqslant Y$ and not $Y \cdot\!\geqslant X$," and $X \doteq Y$ means "$X \cdot\!\geqslant Y$ and $Y \cdot\!\geqslant X$."

DEFINITION 3. (a) $X \geqslant_C Y$ means that $|X| \geqslant |Y|$, i.e., that the cardinal number of X is as large as the cardinal number of Y.

(b) $X \geqslant_D Y$ means that $\dim X \geqslant \dim Y$; this relation is defined only when both X and Y are separable metric spaces.

(c) $X \geqslant_F Y$ (for "Frechet size") means that Y can be embedded homeomorphically in X, i.e., there is a subspace X' of X for which $X' \cong Y$.

(d) $X \geqslant_{LF} Y$ means that each neighborhood in Y can be embedded in X; i.e., if N is a neighborhood in Y, then there is a subspace X' of X for which $X' \cong N$.

Figure 1 depicts the relationships among the quasi-orderings we have defined. It is straightforward to verify each of the implications in Fig. 1, and it is easy to give examples which demonstrate that none of the impli-

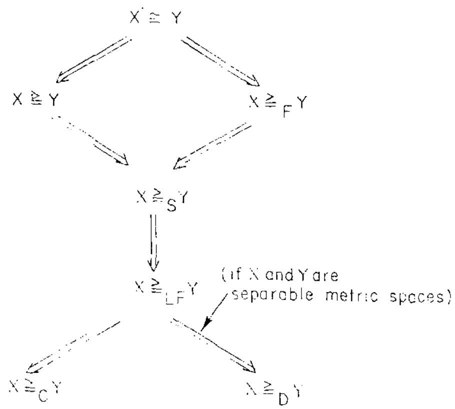

FIGURE 1

cations is reversible. However, when Y is a finite-dimensional Euclidean space, then the statements $X \geqslant_F Y$, $X \geqslant_S Y$, and $X \geqslant_{LF} Y$ *are* all equivalent to one another, because each neighborhood of such a space Y contains a subset which is homeomorphic to Y.

But Theorem 35 of [MR] demonstrates that if (μ, g) is an admissible procedure which uses a space M to realize the competitive outcome function, and if M is Hausdorff, then $M \geqslant_{LF} \overline{M}$.[8] Since $\overline{M}$ is homeomorphic to $\mathbf{R}^{(n-1)(l-1)}$, the observation in the preceding paragraph implies that the theorem's conclusion could equivalently be $M \geqslant_F \overline{M}$ or $M \geqslant_S \overline{M}$. Using the stronger of the two characterizations of size, we have the following theorem.

[8] Hiroaki Osana has pointed out to me that the proof of Theorem 35 only establishes that $M \geqslant_{LF} \overline{M}$, not that the two spaces are locally homeomorphic.

THEOREM 1. *Let (μ, g) be a privacy-preserving procedure which realizes the competitive outcome function, and which uses the message space M. If M is Hausdorff, then $M \geqslant_F \overline{M}$.*

By adopting a different (indeed, a more appealing) definition of informational size than $\cdot\geqslant$, we have recovered the global efficiency result, and have shown it to be equivalent to the local result (in particular, no continuity condition on μ is required, beyond what is embodied in admissibility; hence, we have an even stronger theorem than the (incorrect) one we began with).[9]

We will see in a moment that, in a certain natural sense, the *only* way in which we can recover the global result is to adopt $\geqslant_F$, or some weaker ordering, as our definition of informational size. First, however, let us consider the following example, which shows that we cannot dispense with the condition that M be Hausdorff; the example will also serve to motivate what follows.

EXAMPLE 1. Let A be the simplex S, with the cofinite topology (the proper closed subsets of A are precisely the finite subsets) instead of the usual (Euclidean) topology. Let (μ, g) be the procedure which differs from $(\bar{\mu}, \bar{g})$ only in substituting the space A for S in its message space. In other words, where $\bar{\mu}$ maps E^n onto $\mu(E^n) = \overline{M} \subseteqq S \times \mathbf{R}^{nl}$, we have $\mu(e) = \bar{\mu}(e)$ for each $e \in E^n$ (hence, as sets, $M = \overline{M}$, but they have different topologies); and where $\bar{g}$ maps $\overline{M}$ into $\mathbf{R}^{nl}$, we have g mapping M into $\mathbf{R}^{nl}$, with $g(p, y) = \bar{g}(p, y) = y$, for each $(p, y) \in M$.

The space M is a T_1 space ("points are closed"); it is "almost, but not quite" Hausdorff. It is not true that $M \geqslant_F \overline{M}$; if it *were* true, then a subspace of M (and thus a subspace of $A \times \mathbf{R}^{m-(l-1)}$) would be homeomorphic to $\mathbf{R}^m$, which is clearly impossible. Of course, neither $M \geqslant_S \overline{M}$, $M \cdot\geqslant \overline{M}$, nor $M \geqslant_{LF} \overline{M}$ can be true, either. On the other hand, (μ, g) is quite a reasonable procedure in nearly every other respect: it is privacy preserving; μ is locally threaded; in fact, μ is a continuous function.[10] (μ, g) satisfies very strong conditions, but has a message space which is not quite Hausdorff, and the relation $M \geqslant_F \overline{M}$ fails to hold.

The example is nothing more than a trivial modification of the competitive procedure $(\bar{\mu}, \bar{g})$. Its construction, however, suggests a large class of similar procedures, each one a different variation on $(\bar{\mu}, \bar{g})$: we embed $\overline{M}$ in a "larger" message space M, but otherwise leave the procedure unchanged. The

[9] The condition that M must be Hausdorff was implicitly used in [MR] as well.

[10] The function μ is not locally sectioned, but this is too much to hope for here: Since μ and $\bar{\mu}$ are identical as functions, if μ were locally sectioned then M and $\overline{M}$ would be homeomorphic.

following definition makes precise this notion of embedding a procedure in an "expanded" message space.

DEFINITION 4. Let $(\hat{\mu}, \hat{g})$ and (μ, g) be procedures which use the message spaces $\hat{M}$ and M to realize the function $f: X \to Y$. We say that (μ, g) is an expansion of $(\hat{\mu}, \hat{g})$ if there is an embedding $\psi: \hat{M} \to M$ of $\hat{M}$ into M such that $\mu = \psi \circ \hat{\mu}$.

Notice first that an expansion (μ, g) of a procedure $(\hat{\mu}, \hat{g})$ must use a message space M which is "as large as" $\hat{M}$, in the sense that $M \geqslant_F \hat{M}$. Second, note that, aside from the change in the message space, $(\hat{\mu}, \hat{g})$ and its expansion are identical to one another; the new space M is "larger" (in terms of $\geqslant_F$), but the "new part" of M (i.e., $M \backslash \psi(\hat{M})$) is never used: $\mu(X) = \psi(\hat{\mu}(X)) \subseteq \psi(\hat{M})$. Notice too that (μ, g) preserves privacy if $(\hat{\mu}, \hat{g})$ does.

The following theorem provides extensive possibilities for expansion of the competitive procedure $(\bar{\mu}, \bar{g})$; for example, if A is any neighborhood in $\mathbf{R}^l$, then Theorem 2 guarantees that there is an expansion of $(\bar{\mu}, \bar{g})$ which uses the space $A \times \mathbf{R}^k$ for its message space, where $k = (n-1)(l-1)$. Since $\bar{M}$ is homeomorphic to $S \times \mathbf{R}^k$, Theorem 2 provides us with expansions of $(\bar{\mu}, \bar{g})$ which use message spaces M that are *smaller* than $\bar{M}$, in the sense that $M <\cdot \bar{M}$. The theorem shows, then, that Theorem 31 cannot be recovered unless the ordering $\cdot\!\geqslant$ is abandoned; moreover, it shows that $\geqslant_F$ is a lower bound on the set of relations $\geqslant$ for which Theorem 31 *could* be true. These ideas will be taken up in more detail after the proof of Theorem 2 is described.

THEOREM 2. *Let A be a normal space; if there is a set $S' \subseteq \mathbf{R}^l$ which contains S and which can be embedded as a closed subset of A, then there is an expansion of $(\bar{\mu}, \bar{g})$ which uses the message space $A \times \mathbf{R}^k$, where $k = (n-1)(l-1)$.*

The proof of Theorem 2 involves a rather intricate construction, and will not be given in complete detail here. In the following example, a particular space A is chosen, and an expansion of $(\bar{\mu}, \bar{g})$ is constructed which uses $A \times \mathbf{R}^k$ for its message space. It will then be shown informally how the somewhat simpler construction of the example can be altered to yield expansions for the other spaces A to which the theorem applies.

EXAMPLE 2. Let $k = (n-1)(l-1)$; let $\hat{p} = (0,..., 0, 1) \in \mathbf{R}^k$; let $S' = S \cup \{\hat{p}\}$, with its usual toplogy in $\mathbf{R}^l$; and let $M = S' \times \mathbf{R}^k$. In this example, then, we have $A = S'$; more generally, as in Theorem 2, A can be any normal space into which S' can be embedded as a closed subset. We will construct a procedure (μ, g) which is an expansion of $(\bar{\mu}, \bar{g})$, and which uses

M for its message space; notice that $M <\cdot \bar{M}$. We will first construct an embedding ψ of $\bar{M}$ into M; then we will define μ to be $\psi \circ \bar{\mu}$ (see Fig. 2). Then $\bar{g} \circ \psi^{-1}$ is well defined on $\psi(\bar{M}) \subseteqq M$, and $\bar{g} \circ \psi^{-1} \circ \mu = \bar{g} \circ \bar{\mu} = \rho$, so we need only extend $\bar{g} \circ \psi^{-1}$ to all of M. The only difficult part of this construction is to choose ψ in such a way that $\bar{g} \circ \psi^{-1}$ can be extended to a *continuous* function on M. The intricate part of the example, then, will occur at the beginning, as we build up the function ψ.

$$\begin{tikzcd} E^n \arrow[rr, "\rho"] \arrow[dr, "\bar{\mu}"] \arrow[ddr, bend right, "\mu"'] & & \mathbb{R}^{n\ell} \\ & \bar{M} \arrow[ur, "\bar{g}"] \arrow[d, "\psi"] & \\ & M \arrow[uur, bend right, "g"'] & \end{tikzcd}$$

FIGURE 2

Let $\pi_1' : \mathbf{R}^l \to \mathbf{R}^{l-1}$ be defined by $\pi_1'(x_1, \ldots, x_l) = (x_1, \ldots, x_{l-1})$; for each $x \in \mathbf{R}^l$, we will denote $\pi_1'(x)$ by $\tilde{x}$. The mapping π_1' discards the lth component of each "commodity" vector in $\mathbf{R}^l$.

Let $\pi_1 : \mathbf{R}^{nl} \to \mathbf{R}^{n(l-1)}$ be defined by $\pi_1(x^1, \ldots, x^n) = (\tilde{x}^1, \ldots, \tilde{x}^n)$, where $x^i \in \mathbf{R}^l$ for $i = 1, \ldots, n$. The mapping π_1 discards the lth component of the "commodity" vector of each of the n "households."

Let $\pi_2 : \mathbf{R}^{n(l-1)} \to \mathbf{R}^{(n-1)(l-1)}$ be defined by $\pi_2(\tilde{x}^1, \ldots, \tilde{x}^n) = (\tilde{x}^1, \ldots, \tilde{x}^{n-1})$. The mapping π_2 discards the nth household's allocation.

Let $\pi = \pi_2 \circ \pi_1 : \mathbf{R}^{nl} \to \mathbf{R}^k$.

Let ψ_1 be the restriction to M of the map $\mathbf{id}_S \times \pi : S \times \mathbf{R}^{nl} \to S \times \mathbf{R}^k$. It is relatively easy to verify that ψ_1 is a homeomorphism.

Now let us pause for a moment to consider the case $n = l = 2$, where it is easiest to see what is going on here (Fig. 3). The space M consists precisely

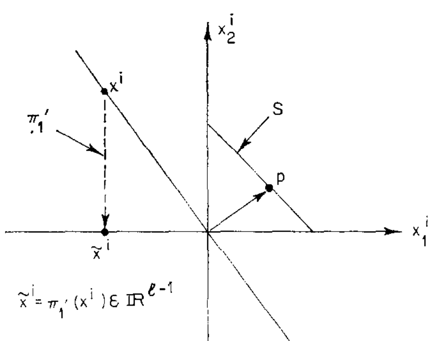

FIGURE 3

of those members (p, x) of $S \times \mathbf{R}^{nl}$ for which $p \cdot x^i = 0$ for each i ($i = 1, \ldots, n$). The mapping π_1 tells us to project each x^i vertically, in household i's "trade space" $\mathbf{R}^2$, onto the $x_1{}^i$ axis. Then π_2 tells us to forget about household 2. Hence, $\psi_1 = \pi_2 \circ \pi_1$ represents each point (p, x) in $\overline{M}$ by the point (p, y) in $S \times \mathbf{R}$, where $y = x_1{}^1$.

Now we return to the general case, where n and l are arbitrary, and we construct the embedding $\psi\colon \overline{M} \to M$ and the procedure (μ, g).

Let **in**: $S \to S'$ be the inclusion map.

Let $\psi_2 = \mathbf{in} \times \mathbf{id}\colon S \times \mathbf{R}^k \to S' \times \mathbf{R}^k$, which is an embedding.

Let $\psi = \psi_2 \circ \psi_1 : \overline{M} \to M$, the embedding toward which we have been working.

Let $\mu = \psi \circ \bar{\mu}$, and let $g\colon M \to \mathbf{R}^{nl}$ be defined as:

$$
\begin{aligned}
g(p, y) &= \bar{g} \circ \psi^{-1}(p, y), && \text{if } p \neq \hat{p}, \\
&= \lim_{p' \to \hat{p}} \bar{g} \circ \psi^{-1}(p', y), && \text{if } p = \hat{p}.
\end{aligned}
$$

We have constructed μ and g in such a way that (μ, g) is an expansion of $(\bar{\mu}, \bar{g})$, *if* (μ, g) is a well-defined procedure. It is obvious that μ is locally threaded; that g is locally sectioned, if well defined; that g is defined and continuous at each $p \neq \hat{p}$; and that g is continuous at each $(\hat{p}, y)$ if it is defined there. Therefore, it remains only to show that $\lim_{p \to \hat{p}} \psi^{-1}(p, y)$ exists for each $y \in \mathbf{R}^k$.

We note first that ψ_2^{-1} does not depend upon p; ψ_2^{-1} simply puts the nth household back in, via $y^n = -\sum_{i=1}^{n-1} y^i \in \mathbf{R}^{l-1}$. Therefore, $\lim_{p \to \hat{p}} \psi_2^{-1}(p, y)$ exists, and we are left to show that $\lim_{p \to \hat{p}} \psi_1^{-1}(p, y)$ exists, for each $y \in \mathbf{R}^{n(l-1)}$. Of course, ψ_1^{-1} simply puts the lth component of each household's allocation back in, by projecting $y^i \in \mathbf{R}^{l-1}$ back up vertically in the trade space $\mathbf{R}^l$, onto the subspace which is orthogonal to p (see Fig. 4). Formally, $\psi_1^{-1}(p, y) = (p, (x^1, \ldots, x^n))$, where each x^i (a member of $\mathbf{R}^l$) is given by

$$
\begin{aligned}
x_j{}^i &= y_j{}^i, && \text{if } j \neq l, \\
&= -(1/p_l) \sum_{j=1}^{l-1} p_j y_j{}^i, && \text{if } j = l.
\end{aligned}
$$

Therefore, $\lim_{p \to \hat{p}} \psi_1^{-1}(p, y) = (p, (x^1, \ldots, x^n))$, where each x^i is given by

$$
\begin{aligned}
x_j{}^i &= y_j{}^i, && \text{if } j \neq l, \\
&= 0, && \text{if } j = l.
\end{aligned}
$$

A proof of Theorem 2 can be developed by altering the construction in Example 2. Let A and S' be as specified in the theorem, and let $\bar{S}$ be the

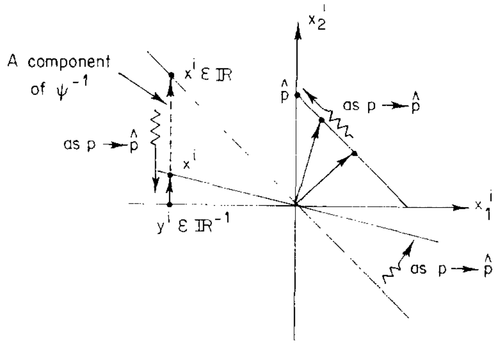

FIGURE 4

closure of S in $\mathbf{R}^l$. Since $\mathbf{R}^{nl}$ is an absolute retract,[11] we can extend any continuous function g': $S' \times \mathbf{R}^k \to \mathbf{R}^{nl}$ to a continuous function g: $A \times \mathbf{R}^k \to \mathbf{R}^{nl}$. Consequently, we need only show that $g \circ \psi^{-1}$: $S \times \mathbf{R}^k \to \mathbf{R}^{nl}$ can be extended to a continuous function g': $S' \times \mathbf{R}^k \to \mathbf{R}^{nl}$, for any set $S' \subseteqq \mathbf{R}^{nl}$ which contains S. It is clearly sufficient, then, to verify the existence of such extensions for just those S' which satisfy $S \subseteqq S' \subseteqq \bar{S}$, and hence for just $S' = \bar{S}$ itself.

Now, the construction in Example 2 will not work for $S' = \bar{S}$. The only points in $\bar{S}$ which case any difficulty here are the first $l - 1$ unit vectors of $\mathbf{R}^l$: if $\bar{p} = \hat{p}$ (the lth unit vector), we have shown that $\lim_{p \to \hat{p}} \psi_1^{-1}(p, y)$ exists; but if $\bar{p}$ is any of the other $l - 1$ unit vectors, the limit does *not* exist (the limit does exist for all other $\bar{p}$ on the boundary of $\bar{S}$). Of course, the problem here is that π_1' projects in a direction which is orthogonal to each of the first $l - 1$ unit vectors; if we define π_1' so as to discard a different component of $\mathbf{R}^l$, then we can pick up the corresponding unit vector, but the limit still fails to exist at the remaining $l - 1$ unit vectors. However, we need only define π_1' in such a way that the direction in which it projects is parallel to some member of S, so that the direction of projection is not orthogonal to *any* member of $\bar{S}$, and then $\lim_{p \to \hat{p}} \psi_1^{-1}(p, y)$ will exist for each $\bar{p} \in \bar{S}$, and we will have our proof.

Now recall our question about Theorem 31 of [MR]: Is there any reasonable restriction on procedures that would render the theorem true? If $l = 2$ (so that S is homeomorphic to the open unit interval $\mathring{I}$), then Theorem 2 guarantees, for example, that if $A = I$ (the closed unit interval), or if $A = C$ (the circle), then $(\bar{\mu}, \bar{g})$ can be expanded to a procedure (μ, g) which uses $A \times \mathbf{R}^k$ for its message space. Now since $S \times \mathbf{R}^k \cong I \times \mathbf{R}^k$ and $S \times \mathbf{R}^k \cong C \times \mathbf{R}^k$ (and $\overline{M} \cong S \times \mathbf{R}^k$), our "reasonable restriction" must rule out both

[11] Equivalently, by Tietze's extension theorem; see, for example, p. 56 of E. H. SPANIER, "Algebraic Topology," McGraw-Hill, New York, 1966.

of those expansions while at the same time admitting $(\bar{\mu}, \bar{g})$. But any expansion of $(\bar{\mu}, \bar{g})$ is identical to it (up to homeomorphism) in every respect, except that it uses a different message space. The restriction we are seeking, then, must either be stated directly in terms of the message spaces themselves (and must exclude such spaces as $I \times \mathbf{R}^k$ and $C \times \mathbf{R}^k$), or else it must require that the entire message space be used (i.e., that $\mu(E^n) = M$). Both kinds of restriction are clearly unsatisfactory, so we must conclude that Theorem 31 simply cannot be recovered. Indeed, one is led by Theorem 2 to the conclusion that, given the definition of a procedure in [1], a relation $\geqslant$ cannot meaningfully be interpreted as an "informational size" comparison unless it satisfies the condition $X \geqslant_F Y \Rightarrow X \geqslant Y$ (certainly it must at least satisfy the condition $M \geqslant_F \overline{M} \Rightarrow M \geqslant \overline{M}$). And this leads one, in turn, to view Theorem 1 as the most fundamental form of the informational efficiency theorem: it demonstrates that we can obtain a very appealing result with precisely the quasi-ordering $\geqslant_F$; i.e., not only is $\geqslant_F$ the infimum of all quasi-orders for which the result is true, the result is true for $\geqslant_F$ as well.

In short, then, it seems to me that the theory that has been developed by Mount and Reiter, together with the analysis that has been presented here, leads one to the view that the most promising (topological) characterization of the informational size of message spaces is provided by the Frechet quasi-ordering $\geqslant_F$. Conversely, the attractiveness of both the result (Theorem 1) and the notion $\geqslant_F$ add credence to the approach that Mount and Reiter have taken, and specifically to the notion of an "informationally admissible" procedure that they have introduced.

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