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The Maximum Theorem and its generalizations[1] have become one of the most useful tools in economic theory. The theorem — first stated and proved, to the best of my knowledge, by Claude Berge [1963, p. 116] — gives conditions under which a "maximizing correspondence" (a correspondence which always picks out the set of maximal elements) will be closed, and hence, in many cases, upper hemicontinuous. A further generalization of Berge's theorem is provided here, one which extends the usefulness of the theorem, especially to equilibrium concepts in multi-person decision situations. The generalization is suggested by a natural decomposition of Berge's result into two parts, and by the nature of the domination concept in game theory. Several applications are suggested, probably none of which is a new result, but each of which becomes transparent in the light of the theorem's general form. Thus, the generalized theorem provides a unified treatment of upper hemicontinuity for the kinds of correspondences which occur frequently in the social sciences.

The original form of the Maximum Theorem was essentially as follows:

THEOREM 1 (*Berge* [1963, p. 116]). *Let* E *and* X *be topological spaces; let* $u\colon E\times Y\to \mathbf{R}$ *be a continuous real-valued function; let* $F\colon E\twoheadrightarrow Y$ *be a continuous and compact-valued correspondence;*[2] *and, for each* $e\in E$, *let* $M(e)=\{y\in F(e)|\forall x\in F(e)\colon u(e, y)\geq u(e, x)\}$. *Then the correspondence* $M\colon E\twoheadrightarrow Y$ *is upper hemicontinuous and compact-valued.*

In applications, E is generally interpreted to be a set of environments, Y a set of actions that might be undertaken, u an objective function (utility, profit, etc.), and $F(e)$ the set of actions actually available in environment e. Then $M(e)$ is the set of optimal actions in environment e.

Notice that $M(e)$ can also be described (for each e) as the intersection of two sets, $M(e)=\mathring{M}(e)\cap F(e)$, where $\mathring{M}(e)$ is simply the set of all actions (whether feasible or not) that are undominated by any action in $F(e)$:

$$\mathring{M}(e) = \{y\in F(e)|\forall x\in F(e)\colon u(e, y) \geqq u(e, x)\}.$$

This suggests that the theorem may be decomposed into properties of $\mathring{M}$ and F. The following proposition verifies that so long as the feasibility correspondence

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[1] For example, Debreu [1969, Theorem 4], Sonnenschein [1971, Theorem 6], and Hildenbrand [1974, p. 29, Theorem 3].

[2] See the Appendix for definitions of correspondences and their properties. Products of topological spaces are always the usual topological product.

F is upper hemicontinuous and compact-valued, then the theorem hinges precisely on the closedness of $\mathring{M}$, the "undominated-set correspondence."[3]

Proposition 1 (*Berge* [1963, p. 112]). *Let* $\mathring{M}: E \twoheadrightarrow Y$ *and* $F: E \twoheadrightarrow Y$ *be correspondences; if* (i) $\mathring{M}$ *is closed, and* (ii) F *is uhc and compact-valued, then* $\mathring{M} \cap F$ *is uhc and compact-valued.*

Once our attention is concentrated upon the correspondence $\mathring{M}$, the "undominated set" terminology suggests that the theorem might apply to game theoretic situations just as well as to individual decision-making. The concept of domination in game theory, however, generally involves "dominating" and "dominated" entities which are different in kind from one another; for example, the dominated entities might be outcomes, or strategy specifications for every player, and the dominating entities the strategies available to the various coalitions. Thus, before the theorem can be applied in such instances, it must be generalized in several ways. Theorem 2 is just such a generalization; in it, we can think of E again as a set of environments; and we can think of Y as a set of possible "proposals" for action; of X as a set of "counterproposals," or "blocking actions;" of $G(e, y)$ as the counterproposals which are actually available in the instance (e, y); and of $D(e)$ as a dominance or blocking relation. Then $\mathring{M}(e)$ is the set of undominated or unblocked actions when e is the environment, and Theorem 2 gives conditions under which the correspondence $\mathring{M}: E \twoheadrightarrow Y$ is closed. (To see that Theorem 2 is indeed a generalization of Theorem 1, let $X = Y$, let G be independent of y, and let $F(e) = G(e, y)$).

Theorem 2. *Let* E, X, *and* Y *be topological spaces, and let* $G: E \times Y \twoheadrightarrow X$ *and* $D: E \twoheadrightarrow X \times Y$ *be correspondences which satisfy the conditions*

(i) G *is lower hemicontinuous, and*

(ii) D *is open*

then the correspondence $\mathring{M}: E \twoheadrightarrow Y$ *is closed, where* $\mathring{M}$ *is defined by* $\mathring{M}(e) = \{y \in Y \mid \forall x \in G(e, y): (x, y) \notin D(e)\}$. *If* $F: E \twoheadrightarrow Y$ *is upper hemicontinuous and compact-valued, then so is* $\mathring{M} \cap F$.

Proof. The last sentence of the theorem follows from Proposition 1. To show that $\mathring{M}$ is closed, let $e_0 \in E$ and $y_0 \in Y$ be such that $y_0 \notin \mathring{M}(e_0)$. We must show that there are neighborhoods U of e_0 and W of y_0 for which $e \in U = \mathring{M}(e) \cap W = \emptyset$. Since $y_0 \notin \mathring{M}(e_0)$, there is an $x_0 \in G(e_0, y_0)$ for which $(x_0, y_0) \in D(e_0)$. Since $\{(e, x, y) \mid (x, y) \in D(e)\}$ is open, there are neighborhoods U_1 of e_0, V of x_0, and W_1 of y_0 such that $(e, x, y) \in U_1 \times V \times W_1 \Rightarrow (x, y) \in D(e)$. Because G is lower hemicontinuous, the set $G^{-1}(V) = \{(e, y) \in E \times Y \mid G(e, y) \cap V \neq \emptyset\}$ is open; because $G(e_0, y_0) \cap V \neq \emptyset$ (the set contains x_0), $G^{-1}(V)$ is a neighborhood of (e_0, y_0). Let U_2 and W_2 be neighborhoods of e_0 and y_0 such that $U_2 \times W_2 \subseteq$

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[3] Berge used this decomposition in his proof of Theorem 1. Note, by the way, that this decomposition has nothing to do with the fact that closedness is equivalent to upper hemicontinuity for a correspondence with a compact range.

$G^{-1}(V)$. Finally, let $U = U_1 \cap U_2$, and let $W = W_1 \cap W_2$.

We now show that U and W are the desired neighborhoods. Let $e \in U$ and $y \in W$; we must show that $y \notin \mathring{M}(e)$. Because $(e, y) \in U_2 \times W_2$, we have $G(e, y) \cap V \neq \emptyset$; let $x \in G(e, y) \cap V$. Since $e \in U_1$, and $x \in V$, and $y \in W_1$, then we have $(x, y) \in D(e)$, which together with $x \in G(e, y)$, implies that $y \notin \mathring{M}(e)$.

Notice that the theorem does not require that the relations $D(e)$ have any order properties, such as irreflexivity or transitivity; indeed, such properties are meaningless here, because $D(e)$ need not even be a relation between the same kinds of objects: X and Y are not in general the same set.[4]

As an example[5] of an application of Theorem 2, let us consider a very abstract social process, in which we have a set A of agents and a collection $\mathscr{S}$ of subsets of A. For each $a \in A$, let X_a be a set of possible actions that the agent a might undertake, and for each $S \in \mathscr{S}$, let X_S denote the product $\prod_{a \in S} X_a$. For each such set S, we suppose that there are correspondences $G_S\colon E \times X_A \twoheadrightarrow X_S$ and $D_S\colon E \twoheadrightarrow X_S \times X_A$, which we interpret as "feasibility" and "dominance" correspondences, just as we did in the paragraph preceding Theorem 2. Consequently, for each $S \in \mathscr{S}$ and each $e \in E$, the set

$$\mathring{M}_S(e) = \{y \in X_A \mid x_S \in G(e, y) \Longrightarrow (x_S, y) \notin D_S(e)\}$$

is the set of all those joint actions which the "coalition" S cannot unilaterally improve upon. The correspondence $\mathring{M} = \bigcap_{S \in \delta} \mathring{M}_S\colon E \twoheadrightarrow X_A$ is thus a kind of equilibrium correspondence: for each $e \in E$, $\mathring{M}(e)$ is the set of undominated actions, or social equilibria. Hence, the following corollary can be interpreted as giving conditions under which the set of social equilibria will vary upper hemicontinuously with changes in the environment.

COROLLARY 1. *Let E be a topological space; let A be a set, and for each $a \in A$, let X_a be a topological space. Let $\mathscr{S}$ be a collection of subsets of A, and for each $S \in \mathscr{S}$, let $X_S = \prod_{a \in S} X_a$, and let $G_S\colon E \times X_A \twoheadrightarrow X_S$ and $D_S\colon E \twoheadrightarrow X_S \times X_A$ be correspondences which satisfy*

(i) G_S is lower hemicontinuous, and

(ii) D_S is open.

Then the correspondence $\mathring{M}$ defined above is closed. If $F\colon E \twoheadrightarrow Y$ is upper hemicontinuous and compact-valued, then so is $M = \mathring{M} \cap F$.

There are several interesting special cases of Corollary 1. Suppose, for example, that for each $a \in A$ and each $e \in E$, there is a relation $P_a(e) \subseteq X_A \times X_A$, which we will interpret as agent a's preference relation over the joint actions in

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[4] For conditions under which M is non-empty-valued, in the case $X = Y$ (in other words, for conditions under which open relations have maximal elements), see Walker [1978] or Bergstrom [1975, pp. 403–404], and the much deeper Sonnenschein [1971, Chap. 10] and Shafer and Sonnenschein [1975, pp. 345–348].

[5] An example in which $X \neq Y$; applications when $X = Y$ are well-known.

X_A. Let $x_a \wedge y$ denote the member z of X_A which is obtained from $y \in X_A$ by replacing y_a with x_a (i.e., $z_a = x_a$, and if $a' \neq a$, then $z_{a'} = y_{a'}$); and define $D_a\colon E \twoheadrightarrow X_a \times X_a$ by $(x_a, y) \in D_a(e) \Leftrightarrow (x_a \wedge y, y) \in P_a(e)$. Now if $\mathscr{S}$ consists of just the singleton sets $\{a\}$, and if $F\colon E \twoheadrightarrow X_A$ is defined by $F(e) = \{y \in X_A | \forall a \in A\colon y_a \in G_a(e, y)\}$, then $M(e)$ is the set of Nash-equilibria in environment e (more precisely, the set of "social equilibria," or generalized Nash-equilibria, of Debreu [1952]). Hence, Corollary 1 tells us that the set of (generalized) Nash-equilibria varies upper hemicontinuously as the environment changes, so long as each preference P_a is open in $E \times X_A \times X_A$ and each feasibility correspondence G_a is continuous and compact-valued.[6]

The core of a finite exchange economy provides another special case of Corollary 1. Suppose that there are l goods, that A is finite, and that an economy $e \in E$ is described by an initial endowment $\omega_a \in \mathbf{R}^l_+$ and a preference $P_a \subseteq \mathbf{R}^l_+ \times \mathbf{R}^l_+$, for each agent $a \in A$; let $\mathscr{P}$ denote the set of all preferences that we will admit for consideration. For each set $S \in \mathscr{S}$, we define the correspondence G_S and D_S as follows (where e denotes the economy $((\omega_a), (P_a)) \in \mathbf{R}^{lA}_+ \times \mathscr{P} = E$):

$$G_S(e, y) = \{x_S \in X_S | \sum_{a \in S} x_a \leqq \sum_{a \in S} \omega_a\},$$

$$D_S(e) = \{(x_S, y) \in X_S \times X_A | (x_a, y_a) \in P_a, \forall a \in S\}.$$

If $\mathscr{S}$ is the collection of all non-empty subsets of A and if $F(e) = G_A(e, y)$ for each e, then $M(e)$, as it is defined in Corollary 1, is the core of the economy e. Each of the correspondences G_S is clearly continuous and compact-valued, and it is easy to show that each D_S is open if the topology on $\mathscr{P}$ is such that the set $\{(x, y, P) | (x, y) \in P\}$ is open in $\mathbf{R}^l_+ \times \mathbf{R}^l_+ \times \mathscr{P}$. Hence, Corollary 1 tells us that the condition we have just given for the topology on $\mathscr{P}$ is sufficient to guarantee that the core of a finite exchange economy will vary upper hemicontinuously with respect to changes in the data which describe the economy.[7]

Theorem 2 can be used to give equally concise proofs that the set of Pareto-optimal actions, the set of voting outcomes, and other such "equilibrium sets," vary upper hemicontinuously, when those concepts are defined appropriately. And of course, previous applications of the Maximum Theorem to individual choice correspondences, such as to demand and supply behavior, can be performed in the same way with Theorem 2 as with Theorem 1 (or previous generalizations of it), since the latter is a special case of Theorem 2.

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[6] Nash [1950, pp. 48–49] and Shafer and Sonnenschein [1975, pp. 345–348] have given conditions under which M is non-empty-valued — i.e., under which Nash equilibria will exist.

[7] Scarf [1971, pp. 169–181] has given conditions under which M will be non-empty-valued — i.e., under which the core of an economy will be non-empty. Other such conditions are, of course, ones under which a market equilibrium exists.

APPENDIX

A correspondence C from a set X to a set Y (denoted $C: X \twoheadrightarrow Y$) is a subset of $X \times Y$. Equivalently, we can think of C as an assignment of a subset $C(x)$ of Y to each member $x \in X$, and of the corresponding subset of $X \times Y$ as the graph of C. A correspondence $C: X \twoheadrightarrow Y$ is non-empty-valued, or compact-valued, etc., if, for each $x \in X$, $C(x)$ is non-empty or compact, etc.

If X and Y are topological spaces, $X \times Y$ their (topological) product, and $C: X \twoheadrightarrow Y$ a correspondence from X to Y, then

(i) C is open if it is an open subset of $X \times Y$;
(ii) C is closed if it is a closed subset of $X \times Y$;
(iii) C is lower hemicontinuous if, for each $x_0 \in X$ and each open set $V \subseteq Y$ which intersects $C(x_0)$, there is a neighborhood U of x_0 for which $x \in U \Rightarrow C(x) \cap V \neq \emptyset$;
(iv) C is upper hemicontinuous if, for each $x_0 \in X$ and each open set $V \subseteq Y$ which contains $C(x_0)$, there is a neighborhood U of x_0 for which $x \in U \Rightarrow C(x) \subseteq V$;
(v) C is continuous if it is both upper and lower hemicontinuous.

See Berge [1963, Chapter 6] or Hildenbrand [1974, pp. 21–35] for more detailed treatments of correspondences and their topological properties. Note, however, that in contrast to Hildenbrand, a correspondence need not be non-empty-valued here; and in contrast to Berge, a correspondence need not be compact-valued here in order to be upper hemicontinuous.

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