Time, Uncertainty, and Incomplete Markets

9.1 Suppose half the people in the economy choose according to the utility function

$$u^A(x_0, x_H, x_L) = x_0 + 5x_H - .3x_H^2 + 5x_L - .2x_L^2$$

and the other half according to the utility function

$$u^B(x_0, x_H, x_L) = x_0 + 5x_H - .1x_H^2 + 5x_L - .2x_L^2$$

where

x_0 represents consumption "today,"
x_H represents consumption "tomorrow" in event H, and
x_L represents consumption "tomorrow" in event L.

Storage of the consumption good from today until tomorrow is not possible. Each person is endowed with twelve units of the good in each of the two periods, no matter which of the two possible events occurs.

In your answers, consider only allocations that give all type A people the same consumptions and all type B people the same consumptions, so that you will be able to completely describe an allocation with the six variables $x_0^A, x_H^A, x_L^A, x_0^B, x_H^B$, and x_L^B.

(a) Which allocations are Pareto optimal?

(b) Determine the Arrow-Debreu equilibrium — *i.e.*, the Arrow-Debreu prices and allocation.

(c) Suppose that the only market is a credit market (*i.e.*, a market for borrowing and lending). There are no markets in which one can insure oneself against either of tomorrow's two possible events. What will be the competitive equilibrium interest rate and how much will each person borrow or save? Is the equilibrium allocation Pareto optimal?

(d) In addition to the credit market in (c), suppose there is another market as well, in which one can buy or sell insurance today against the occurrence of event H. Each unit of insurance that a person purchases is a contract in which the seller of the contract agrees to pay the buyer one unit of consumption tomorrow if event H occurs. Let p denote the market price of the insurance: the buyer pays the seller p units of consumption today for each unit of insurance he purchases. Determine the competitive equilibrium prices (*i.e.*, the interest rate and the price p of insurance) and the equilibrium allocation.

9.2 Alice and Bill each have fifteen dollars today, and each will also have fifteen dollars tomorrow. Before tomorrow arrives an election is going to take place. Bill knows that if the Democrats win the election there will be lots of parties with lots of celebrities; because he's such a party animal, Bill would like to have more money in the event that the Democrats win, in order to enable him to attend all the parties. Alice is an economist and does not attend parties, so her intertemporal preferences do not place as much weight on the event that the Democrats win. Specifically, Alice's and Bill's intertemporal utility functions are

$$u_A(x_{A0}, x_{AD}, x_{AR}) = x_{A0} + 9x_{AD} - .4x_{AD}^2 + 12x_{AR} - .4x_{AR}^2$$
$$u_B(x_{B0}, x_{BD}, x_{BR}) = x_{B0} + 9x_{BD} - .2x_{BD}^2 + 12x_{BR} - .4x_{BR}^2 \ ,$$

where x_{i0} denotes dollars consumed today by i, and $x_{i\theta}$ denotes dollars consumed tomorrow by i in state θ.

(a) Determine the Arrow-Debreu prices and allocation(s) for the economy consisting of just Alice and Bill. What is the interest rate?

(b) Suppose the only markets open today in which one can contract for dollars tomorrow are two security markets. Security Gamma returns one dollar tomorrow in each state; Security Delta returns one dollar tomorrow if the Democrats win, but requires the holder to pay a dollar tomorrow if the Republicans win. (Gamma securities are generally sold by banks; Delta securities are generally sold by bookmakers.) What are the equilibrium prices (today) of these two securities? How many of each will Alice and Bill buy?

9.3 You're teaching an undergraduate intermediate economics course and you must design a lecture on uncertainty and insurance markets. Your class has already learned about the concept of general equilibrium of markets and about Pareto efficiency, and you've used the Edgeworth box device to help teach these ideas. Describe the simplest possible model you could use (two people, one good, two possible states of the world, no consumption before the uncertainty is resolved) to demonstrate that the market outcomes will generally be inefficient if there are uncertainties for which no markets exist for individuals to "trade risk" with one another.

9.4 Either the Republicans or the Democrats will win the next election — *i.e.*, one of the two states of the world $\theta = R$ or $\theta = D$ will occur. Apu and Bart are each endowed with ten pesos today; each will also be endowed, for certain, with fifteen pesos tomorrow. Each one's preferences are described by a von Neumann-Morgenstern utility function of the form

$$u(c_0, c_R, c_D) = c_0 + E(5 \log c_\theta) = c_0 + 5\pi_R \log c_R + 5\pi_D \log c_D,$$

where c_0 denotes pesos consumed today; c_θ denotes consumption of pesos in state θ; and π_θ denotes the individual's subjective probability assessment that state θ will occur. Apu believes that the two states are equally likely to occur, but Bart believes there is a 3/4 chance that the Republicans will win.

(a) Determine the interior allocations that are Pareto optimal.

For parts (b), (c), and (d) assume that Apu and Bart are the only two traders and that each one behaves as a price-taker in all markets.

(b) Assume that the only market available is a borrowing and lending market. What will the equilibrium interest rate be, and how much will each person save or borrow? How would your answers change if the individuals' subjective probabilities were different?

(c) Assume that there are complete Arrow-Debreu contingent claims markets. Determine the equilibrium prices and consumption levels. What is the implicit interest rate?

(d) Now suppose that the only markets open are a borrowing and lending market (in which contracts are not state-contingent) and an insurance market for state D: in this market insurance contracts that will pay off one peso tomorrow if the Democrats win can be bought and sold; the premium (the price of a one-peso contract) is p pesos, to be paid today. What will the equilibrium interest rate and premium be, how much will each individual save or borrow, and how much insurance will each one buy or sell?

9.5 Andy's income today is \$20 per unit of time (*e.g.*, per hour). If universal health care legislation is passed within the next year, then his income tomorrow will be \$20; but if the legislation fails to pass, his income tomorrow will be only \$10. Beth sells insurance, and her income today is also \$20. If health care legislation is passed, her income tomorrow will be \$10, but if the legislation fails to pass, her income tomorrow will be \$20. Andy's preferences are described by the utility function

$$u^A(x_0, x_H, x_F) = x_0 + 5\log x_H + 6\log x_F$$

and Beth's by the function

$$u^B(x_0, x_H, x_F) = x_0 + 10\log x_H + 3\log x_F,$$

where x_0 denotes the individual's spending today, x_H denotes spending tomorrow if the legislation passes, and x_F denotes spending tomorrow if the legislation fails to pass (all measured in the same units).

(a) Determine the Pareto efficient allocation(s).

(b) Determine the Arrow-Debreu allocation(s) and prices.

(c) Suppose the only markets are spot markets and a credit market. Is the equilibrium allocation Pareto efficient (and how do you know this)? Do not attempt to find the equilibrium interest rate, spot prices, or allocation.

In (d), (e), and (f) you can solve directly, or you can appeal to the complete-markets security pricing formula.

(d) In the Arrow-Debreu market structure, what is the (implicit) interest rate?

(e) Suppose the only securities are shares in the firm Gamma Technologies and shares in the firm Delta Insurance. Each share of Gamma will yield \$2 if the legislation passes and \$1 if the legislation fails. Each share of Delta will yield \$1 if the legislation passes and \$2 if it fails. Determine the equilibrium security prices and Andy's and Beth's holdings of securities.

(f) In the market structure in (e), what portfolio would one have to hold in order to guarantee oneself a return of \$1 tomorrow, whether the legislation passes or not? What would be the cost of the portfolio? What would you say is the interest rate, and why?

9.6 Ann currently has five dollars per hour to spend and Bev has fifteen dollars per hour. In a few years Bev will have to retire; then she will have only four dollars per hour to spend, but then Ann will have sixteen dollars per hour. Both women are making their plans under some uncertainty: they don't know whether the Republicans or the Democrats will be in power when Bev retires. They make their plans based on preferences described by the following utility functions, where x_0 denotes consumption today, x_R denotes consumption tomorrow (*i.e.*, after Bev retires) in the event that the Republicans are in power, and x_D denotes consumption tomorrow if the Democrats are in power, and where each x_θ is measured in dollars per hour:

$$u^A(x_0, x_R, x_D) = x_0 x_R^{\frac{1}{2}} x_D^{\frac{1}{3}}\ ; \qquad \text{i.e.,} \quad MRS^A = \frac{x_0^A}{2x_R^A} \quad \text{and} \quad MRS^A = \frac{x_0^A}{3x_D^A}$$

$$u^B(x_0, x_R, x_D) = x_0 x_R^{\frac{1}{3}} x_D^{\frac{1}{2}}\ ; \qquad \text{i.e.,} \quad MRS^B = \frac{x_0^B}{3x_R^B} \quad \text{and} \quad MRS^B = \frac{x_0^B}{2x_D^B}$$

(a) There is an Arrow-Debreu equilibrium for Ann and Bev in which they each consume ten dollars per hour today. Determine the Arrow-Debreu prices and the women's state-dependent consumptions.

(b) Suppose that the only market for intertemporal trade is the market for a bond that costs one dollar per hour today and pays off $1 + r$ dollars per hour tomorrow, no matter which party is in power. Determine the equilibrium rate r and the state-dependent consumptions, assuming both women behave as price-takers. (*Hint*: This equilibrium also involves Ann and Bev each consuming ten dollars per hour today.)

(c) Is the equilibrium in (b) Pareto efficient? Explain.

(d) Suppose there are four securities being traded today: Security #1 pays one dollar per hour if the Republicans are in power and nothing if the Democrats are in power. Security #2 pays one dollar per hour if the Democrats are in power and nothing if it is the Republicans. Security #3 is a bond that pays twelve dollars per hour for certain. Security #4 pays the holder \$60 per hour if the Republicans are in power and requires the holder to pay \$12 per hour if the Democrats are in power. Using y_k to denote the number of units of Security #k that she buys, write down the constraints that Ann's choices must satisfy.

(e) Assuming price-taking behavior, what must the equilibrium prices of the four securities in (d) be today (and why), and what will be Ann's and Bev's equilibrium consumption streams?

9.7 The only chips that exist today are X-chips; there are only two X-chips, and Mr. B owns them both. Today's X-chips will perish by tomorrow (any chips not consumed today are wasted), but tomorrow there will again be two X-chips and again Mr. B will own them both. But tomorrow may turn out to be a "high-tech" tomorrow (sometimes referred to as "state H"), in which case there will also be two Y-chips (which will be extremely powerful), and Mr. A will own both of them. If, alas, tomorrow turns out to be "low-tech" (sometimes referred to as "state L"), then there will not be any Y-chips. Mr. A and Mr. B make up the entire economy; each has the same preferences, described by the utility function

$$u(x_0, x_L, x_H, y) = x_0 + (1 - \pi) \ln x_L + \pi \ln x_H + 6\pi \ln y,$$

where π is the subjective probability he places on the event that tomorrow will turn out to be high-tech, and where

x_0 denotes his consumption of X-chips today,

x_L denotes his consumption of X-chips in a low-tech tomorrow,

x_H denotes his consumption of X-chips in a high-tech tomorrow, and

y denotes his consumption of Y-chips in a high-tech tomorrow.

Each man believes there is a one-third chance that tomorrow will turn out to be high-tech.

(a) Determine the set of all Pareto optimal allocations. How would your answer be changed if both men were wrong — specifically, what if each believes the probability of a high-tech tomorrow is one-third, but it is actually one-half?

(b) Determine an Arrow-Debreu price-list for contingent claims on all goods.

(c) Now suppose that the only markets that are open today are a spot market for today's X-chips (on which the price is $p_0 = 1$), and two securities markets, H and L. A unit of security θ ($\theta = H, L$), which can be purchased for ψ_θ dollars today, will return one dollar if (and only if) state θ occurs. If tomorrow turns out to be high-tech, spot markets for the two kinds of chips will be open, with prices q_x and q_y. Of course, if tomorrow turns out to be low-tech, there will be only one good, so no trade will take place in that event. It happens that there is a rational expectations equilibrium in which the spot prices tomorrow are $(q_x, q_y) = (1, 6)$. Determine the rest of the equilibrium – *i.e.*, the security prices (ψ_H, ψ_L), the quantity of each security purchased or sold by each individual, and the individual consumption levels. Indicate how one can be assured that the equilibrium you have found is indeed an equilibrium.

9.8 Today Anne and Beth are young and productive: they are endowed with, respectively, $\mathring{x}_{A0}$ and $\mathring{x}_{B0}$ units of the economy's all-purpose consumption good, simoleans. Each of them may, with some probability, live long into her retirement years. Denote the four possibilities, or "states," by YY (both survive), NN (neither survives), YN (Anne alone survives), and NY (Beth alone survives). For each of the four states θ, let π_θ denote the probability that the state will occur. Each woman's endowment in her retirement years, if she survives, will be $\mathring{x}_{A1}$ (for Anne) and $\mathring{x}_{B1}$ (for Beth). (Note that each one's old-age endowment is independent of whether the other survives.) Anne's and Beth's preferences for alternative consumption plans are described by the following utility functions, where $x_{i\theta}$ denotes person i's consumption in state θ and x_{i0} denotes person i's consumption today:

$$
\begin{aligned}
u(x_{A0}, x_{AYY}, x_{AYN}) &= \alpha x_{A0} + \pi_{YY} \log x_{AYY} + \pi_{YN} x_{AYN} \\
u(x_{B0}, x_{BYY}, x_{BNY}) &= \alpha x_{B0} + \pi_{YY} \log x_{BYY} + \pi_{NY} x_{BNY} \ .
\end{aligned}
$$

Express your answers to the following questions in terms of the parameters that describe the economy – *i.e.*, in terms of α, β, the endowments, and the probabilities. Assume that Anne and Beth can exchange goods only among themselves.

(a) Determine which allocations are Pareto efficient.

(b) Determine the Arrow-Debreu equilibrium prices and consumptions. (In other words, assume there are markets for deliveries that are contingent on the relevant states occurring; Anne and Beth are the only participants in these markets, and they are price-takers.)

(c) In (a) and (b) your answers should obviously not have depended upon the probability π_{NN}; but they should also not have depended upon the probabilities π_{YN} and π_{NY}. Also, even if the women's old-age endowments were not independent of the other's survival, your answer would not have depended upon either $\mathring{x}_{AYN}$ or $\mathring{x}_{BNY}$. Why are the answers independent of any parameters involving the states YN and NY, and how general is this result?

(d) Suppose the women's beliefs about their survival probabilities were not the same – *i.e.*, denote Anne's beliefs by π^A_θ and Beth's by π^B_θ for each of the four states θ, and suppose that $\pi^A_{YY} \neq \pi^B_{YY}$. How would this change your answers in (a) and (b)? In particular, would an equilibrium now exist only for very special parameter values; would an equilibrium never exist; would equilibria (when they exist) no longer be Pareto efficient?

9.9 The economy is endowed today with $\mathring{x}_0$ bushels of corn, the only commodity anyone cares about. There will be no endowment tomorrow, but it is possible to grow corn for tomorrow by planting some of today's endowment today. There is some uncertainty about what the growing conditions will be during the intervening period: if conditions turn out to be Good, then each bushel planted will yield α_G bushels tomorrow; if conditions instead turn out to be Bad, then each bushel planted will yield only α_B bushels tomorrow. There are n households in the economy, and each household's preferences are representable by a continuously differentiable utility function $u_i(x_0^i, x_B^i, x_G^i)$, where x_B^i and x_G^i denote the household's consumption of corn tomorrow in states B and G (*i.e.*, under Bad conditions and under Good conditions).

(a) Derive the marginal conditions (expressed in terms of households' marginal rates of substitution) that characterize the interior Pareto efficient allocations. ("Derive" means to show how you obtained the conditions.)

For the remainder of this question, assume that $\alpha_B = 1$ and $\alpha_G = 3$; that there are only two households, labeled a and b; and that their utility functions are

$$u^a(x_0, x_B, x_G) = x_0 + x_B - \tfrac{1}{6}x_B^2 + x_G - \tfrac{1}{36}x_G^2$$
$$u^b(x_0, x_B, x_G) = x_0 + x_B - \tfrac{1}{6}x_B^2 + x_G - \tfrac{1}{18}x_G^2 \ .$$

(b) Determine all the Pareto efficient allocations.

(c) Determine the Arrow-Debreu prices for contingent claims.

(d) Describe an alternative market structure in which, instead of contingent claims for future delivery of goods, what is traded today is securities that return dollar amounts tomorrow, and describe conditions on the securities that are sufficient to ensure that the market equilibrium will be Pareto efficient. Explain why efficiency is achieved under these conditions. If one of the securities is a bond that sells for one dollar today and returns 1+r dollars tomorrow, what will be the value of r in such an efficient equilibrium?

9.10 The economy has an endowment of corn. The corn can be allocated to consumption this year and to planting, which will yield corn next year. Each bushel planted this year will yield ψ_H bushels next year if rainfall is High during the intervening months, and ψ_L bushels if rainfall is Low. No one looks farther ahead than next year: each consumer has a utility function in which the only arguments are x_0, x_H, and x_L (consumption this year; consumption next year if rainfall is High; and consumption next year if rainfall is Low). Everyone's utility function is differentiable. Let z denote the number of bushels that are planted this year, and let MRS_H and MRS_L denote an individual's marginal rates of substitution between consumption next year (x_H or x_L) and consumption today (x_0).

(a) Determine the marginal conditions that characterize the Pareto efficient interior allocations.

(b) Show that the Arrow-Debreu (complete contingent-claims markets) equilibrium is Pareto efficient by showing that it satisfies the marginal conditions you've derived in (a) and the constraint satisfaction conditions. (It may be helpful to remember that if production has constant returns to scale, then a Walrasian equilibrium must yield zero profit to producers.)

(c) Assume that $\psi_H = 3$ and $\psi_L = 2$, and that there are two consumers, labeled A and B, whose endowments are $(\mathring{x}_0^A, \mathring{x}_H^A, \mathring{x}_L^A) = (20, 2, 2)$ and $(\mathring{x}_0^B, \mathring{x}_H^B, \mathring{x}_L^B) = (10, 13, 8)$, and whose preferences are represented by the utility functions

$$u^A(x_0, x_H, x_L) = x_0 + \frac{1}{3}x_H - \frac{1}{180}x_H^2 + x_L - \frac{1}{24}x_L^2$$

and

$$u^B(x_0, x_H, x_L) = x_0 + \frac{1}{3}x_H - \frac{1}{90}x_H^2 + x_L - \frac{1}{36}x_L^2.$$

Verify that there is an Arrow-Debreu equilibrium in which $z = 5$. Determine the equilibrium prices and consumption plans. Verify that total profit is zero. (Don't forget that the markets are for *contingent claims*.) What is the implicit interest rate?

(d) Are there any other Arrow-Debreu equilibria? Verify your answer.

9.11 There are n consumers, only one commodity, and no production is possible. There is uncertainty about which of two possible events (states of the world) will occur. Let π_i denote consumer i's belief about the probability that state 1 will occur (therefore $1 - \pi_i$ is his belief that state 2 will occur), and let $x_i(1)$ and $x_i(2)$ denote consumption by consumer i in states 1 and 2. Every one of the consumers chooses so as to maximize the expected value (according to his own probability estimate π_i) of the same function, $v(z) = z^\alpha$, where z denotes his consumption level and where $0 < \alpha < 1$. Each consumer's endowment of the commodity is unaffected by the state that occurs.

(a) Explain how a contingent claims market would operate for this economy. Are there gains to be had by exchange of contracts? Why or why not?

(b) Derive consumer i's demand for $x_i(2)$ in terms of the price ratio for contracts and i's endowment, say w_i. How will a change in the price ratio or a change in w_i affect the demand for $x_i(2)$?

(c) Suppose some π_i increases. How is the Walrasian equilibrium changed?

9.12 Each trader i is endowed today with $\mathring{x}_0^i$ simoleans. When tomorrow arrives, each trader will again be endowed with simoleans, but his endowment $\mathring{x}_s^i$ will depend upon s, the state of the world. It's not possible to alter any of the endowments – for example by production, storage, etc. – but the traders can transfer simoleans to one another. Let N denote the (finite) set of traders i, and let S denote the (finite) set of possible states s. Each trader's preference can be described by a utility function of the form

$$u^i(x_0^i, (x_s^i)_{s\in S}) = x_0^i + \sum_{s\in S} \alpha_s^i \log x_s^i.$$

(a) Write down a maximization problem for which the solutions are the Pareto efficient allocations, and give the first-order marginal conditions (FOMC) that characterize the interior solutions.

(b) Use the FOMC to derive the interior Pareto allocations and the Arrow-Debreu prices as functions of the parameters $\mathring{x}_0$, $(\mathring{x}_s)_{s\in S}$, and $(A_s)_{s\in S}$, where $A_s := \sum_{i\in N} \alpha_s^i$ for each $s \in S$.

For the remainder of this problem, let $N = \{\text{Amy,Bill}\} = \{A, B\}$, let $S = \{\text{High,Low}\} = \{H, L\}$, and let

$$\alpha_H^A = 4, \quad \alpha_L^A = 2, \quad \mathring{x}_H^A = 12, \quad \mathring{x}_L^A = 6, \quad \mathring{x}_0^A = 10,$$
$$\alpha_H^B = 2, \quad \alpha_L^B = 4, \quad \mathring{x}_H^B = 12, \quad \mathring{x}_L^B = 6, \quad \mathring{x}_0^B = 10.$$

(c) Determine the Pareto efficient allocations.

(d) Determine the Arrow-Debreu equilibrium — *i.e.*, the Arrow-Debreu prices and allocation.

(e) Suppose the only market is a credit market, in which Amy and Bill can lend simoleans today in exchange for receiving simoleans tomorrow, or alternatively can borrow simoleans today in exchange for a promise to deliver simoleans tomorrow. Write down Amy's utility-maximization problem and derive the first-order conditions that characterize the decision she will make. Express her FOMC as a condition that relates her marginal rates of substitution to the market rate of interest.

(f) In addition to the credit market, suppose there is also a security Gamma that returns two simoleans in state H and three simoleans in state L. Use the Arrow-Debreu pricing formula to determine the equilibrium price of this security and the equilibrium interest rate. How much will each person borrow or lend, and how much of the security Gamma will each person hold?

9.13 Half the people in the economy are Type A personalities and the other half are Type B. Type A personalities all choose according to the utility function

$$u_A(x_0, x_H, x_L) = x_0 + 5x_H - .3x_H^2 + 3x_L - .3x_L^2$$

and Type B personalities all choose according to the utility function

$$u_B(x_0, x_H, x_L) = x_0 + 5x_H - .4x_H^2 + 3x_L - .2x_L^2,$$

where x_0 represents consumption "today," x_H represents consumption "tomorrow" in state H, and x_L represents consumption "tomorrow" in state L.

Each person is endowed with six units of the good today. Type A people will be endowed with four units tomorrow in state H and only two units in state L; Type B people will be endowed with ten units tomorrow in state H and eight units in state L. Storage of the consumption good from today until tomorrow is not possible. The two states are mutually exclusive and exhaustive.

In your answers, consider only allocations that give all Type A people the same state-contingent consumption bundle and all Type B people the same state-contingent consumption bundle. You can therefore describe an allocation with just the six variables $x_{A0}, x_{AH}, x_{AL}, x_{B0}, x_{BH}, x_{BL}$.

(a) Determine the set of interior Pareto allocations.

(b) Determine the Arrow-Debreu prices and allocation.

(c) Assume that the only market is a credit market — a market for borrowing and lending. There are no markets in which one can insure oneself against either of tomorrow's two possible states. What will be the competitive interest rate, and how much will each person borrow or save?

(d) In addition to the credit market in (c), suppose there is another market as well, in which one can buy or sell insurance today against the occurrence of state H. Each unit of insurance that a person purchases is a contract in which the seller of the contract agrees to pay the buyer one unit of consumption tomorrow *if* state H occurs. Let p denote the market price of the insurance: the buyer pays the seller p units of consumption today for each unit of insurance he purchases. Determine the competitive equilibrium prices (i.e., the interest rate and the price p of insurance) and the equilibrium allocation. How much does each person borrow or save, and how much insurance does each person buy or sell?

(e) Are the allocations in (c) and (d) Pareto efficient? Explain.