

Adding Production to the Model

4.1 There are two people (A and B) and two goods (wheat and bread). One production process is available, a process by which one bushel of wheat can be turned into two loaves of bread. The individuals' preferences are described by the utility functions

$$u^A(x, y) = xy \quad \text{and} \quad u^B(x, y) = x^2y.$$

where x is the person's consumption of wheat (in bushels) and y is the person's consumption of bread (in loaves). The two people are endowed with a total of 60 bushels of wheat and no bread. For the consumption allocations in (a), (b), and (c) below, do the following: if the given allocation is Pareto optimal, then verify it; if the given allocation is not Pareto optimal, find a feasible Pareto improvement.

(a) $(x_A, y_A) = (12, 24)$ and $(x_B, y_B) = (24, 24)$.

(b) $(x_A, y_A) = (20, 20)$ and $(x_B, y_B) = (20, 20)$.

(c) $(x_A, y_A) = (20, 40)$ and $(x_B, y_B) = (10, 10)$.

4.2 The economy consists of two people (Mr. A and Mr. B) and two goods (the quantities of which will be denoted by x and y). There is a single production process, which can turn the x -good into the y -good as follows:

The first four units of output can be produced at a (real) marginal cost of one-half input unit for each unit of output;
the next four units at a marginal cost of one input unit for each unit of output;
and remaining units at a marginal cost of two input units for each unit of output.

The total endowment is ten units of the input good (the x -good) and none of the output good (the y -good). Thus, the maximum output possible is ten units. Each consumer's preference is described by the utility function $u(x, y) = xy$.

Consider the following allocation: Each person consumes the bundle $(x, y) = (2, 4)$; eight units of output are produced using six units of input. Is this allocation Pareto optimal? If so, prove it. If not, find a Pareto improvement.

4.3 A small town produces only a single product – apples – for sale in external markets. The town’s resources consist of two orchards (one containing only tall trees and the other containing only short trees) and two kinds of workers (giants and midgets). The technology for producing apples is such that one worker works with one tree, to produce apples according to the following table, which gives the daily apple production from each of the four possible ways that a worker can be combined with a tree:

	Tall Tree	Short Tree
Midget	1	3
Giant	8	4

There are 10 midgets and 20 giants in the town, and there are 40 tall trees and 5 short trees. None of the town’s resources can be used for any other purposes, either inside or outside the town.

(a) What is the efficient allocation of workers to trees? Are any of the resources unemployed in this allocation? Determine the marginal product of each of the four resources in this allocation.

Owners of the trees pay workers a piece rate – i.e., a per-apple wage. Each worker in the Tall Tree Orchard is paid P_T for each apple he picks, and each worker in the Short Tree Orchard is paid P_S per apple. The tree owners sell all apples that are picked; the apples are sold in the external apple market, where the price of an apple is P .

(b) If workers are free to move between orchards, what condition(s) must P_T and P_S satisfy in order to sustain the efficient allocation?

(c) Under competitive conditions, what will be the equilibrium piece rates and what profits (if any) will each of the resource owners earn? In equilibrium, determine whether any of the factor prices differ from the value of the factor’s marginal product.

(d) Now suppose it’s apple pickers who sell the apples in the external market. Each worker hires a tree, paying the tree’s owner for each apple the tree yields: R_T per apple to tall tree owners and R_S per apple to short tree owners. How will the competitive equilibrium differ from the one in (c)?

4.4 There are only two goods in the world, bread and wheat, quantities of which are denoted by x and y , respectively. There are 101 people in the economy, 100 of them called “consumers” and one “producer.”

Each consumer is endowed with 40 units of wheat and no bread and has a preference ordering described by the utility function $u(x, y) = y - (1/2)x^2 + 8x$.

The producer has no endowment of either good, but she is the sole owner of the economy’s only production process, which can turn wheat into bread at the rate of one bread unit for every four wheat units used as input. The producer cares only for wheat; i.e., her preference ordering is described by the utility function $u(x, y) = y$.

The consumers all behave as price-takers, but the producer behaves as a monopolist: the price of wheat is always \$1 per unit, and the producer sets the price p of her product (bread) so as to maximize her resulting utility (i.e., to maximize her dollar profit).

- (a) What price will the monopolist charge for each unit of bread, and how much will each consumer buy?
- (b) If the outcome in (a) is Pareto optimal, then verify it. If it’s not Pareto optimal, find a Pareto optimal allocation that makes all 101 people strictly better off than in (a).
- (c) Determine the consumer surplus, producer surplus, and total surplus in the monopoly outcome in (a) and at the Pareto optimal outcome you identified in (b).

4.5 Andy, Bob, and Cathy each have the same preferences for wine and grapes, described by the utility function $u(x, y) = xy$, where x and y denote an individual’s consumption of wine (y gallons) and grapes (x bushels). Grapes can be turned into wine; it takes three bushels of grapes to produce each gallon of wine. This production process is available to everyone – i.e., everyone has the ability to produce wine from grapes at this rate. Andy and Bob each own 12 bushels of grapes and Cathy owns 24 bushels of grapes. No one owns any wine.

- (a) Determine the Walrasian equilibrium prices, production levels, and consumption bundles.
- (b) Now assume that every bushel of grapes can produce a gallons of wine. Determine the Walrasian equilibrium, and determine the set of all Pareto allocations.

4.6 There are only two goods, grapes and wine. There is a single production process available, which can transform grapes into wine according to the following production function, in which z denotes the pounds of grapes used as input and $f(z)$ the resulting quarts of wine obtained as output:

$$f(z) = \begin{cases} 0, & \text{if } 0 \leq z \leq 20 \\ z - 20, & \text{if } 20 \leq z \leq 80 \\ 20 + \frac{1}{2}z, & \text{if } z \geq 80. \end{cases}$$

There are ten identical consumers, each with a preference ordering described by the utility function $u(x, y) = xy$, where x and y denote pounds of grapes and quarts of wine consumed. Each consumer owns 12 pounds of grapes; there are no other grapes and there is no other wine at all except what is produced.

- (a) Draw the set of all the aggregate consumption bundles (x, y) that are feasible.
- (b) Determine all the Pareto optimal production-and-consumption plans in which each consumer receives the same bundle as every other consumer.
- (c) Consider the plan in which $z = 60$; $(x_{10}, y_{10}) = (24, 4)$; and $(x_i, y_i) = (4, 4)$ for $i = 1, \dots, 9$. Find a Pareto improvement upon this plan.

For questions (d),(e), and (f), assume there is a single firm that owns the production process and the consumers all behave “competitively” – *i.e.*, they are price-takers. Assume that the consumers share equally in any profit that the firm earns.

- (d) Write down the firm’s demand correspondence for grapes, being careful to indicate the price lists for which the firm’s demand is not defined.
- (e) Assume that the price of grapes is three dollars per pound and the price of wine is five dollars per quart. How much will each consumer demand of each good?
- (f) Verify that there is no Walrasian equilibrium.

4.7 The only two goods in the economy are X and Y . Carol is the sole owner of the only firm in the economy, which can turn Y into X according to the production function $q = f(z)$, where $f(z) = 2\sqrt{z}$. Thus, z denotes the amount of Y used as input and q denotes the amount of X produced as output. Carol has no desire to consume any X ; she consumes only Y . There are two other people in the economy, Andy and Bert, whose preferences are described by the utility functions

$$u_A(x_A, y_A) = y_A + 36x_A - \frac{1}{2}x_A^2 \quad \text{and} \quad u_B(x_B, y_B) = y_B + 24x_B - \frac{1}{2}x_B^2,$$

where x_i and y_i denote individual i 's consumption of X and Y . The economy has no endowment of X , but each person owns 600 units of Y . Note that Andy's and Bert's marginal rates of substitution are $MRS_A = 36 - x_A$ and $MRS_B = 24 - x_B$, and that the real marginal cost of production is $(1/2)q$.

- (a) Verify that there is a Pareto efficient production-and-consumption plan in which $q = 30$.
- (b) Are there any other Pareto efficient production-and-consumption plans? If so, find one. If not, verify that there aren't.
- (c) Find a Walrasian equilibrium (i.e., a market equilibrium) in which the firm and all three consumers are price-takers. (Assume that the price of Y is one dollar per unit.) Determine the price of X , the output and input levels, the profit (if any), and the bundles that all three people consume.
- (d) If Carol operates the firm as a monopoly she will charge a price of \$20 for each unit of X she sells. Verify that price, and determine how much she will produce, her profit, and the resulting consumption bundles.
- (e) Find a plan that makes everyone strictly better off than in (d), i.e., a strict Pareto improvement. (The Pareto improvement you find need not be Pareto efficient.)
- (f) Find a Pareto efficient plan that makes everyone strictly better off than in (d).

4.8 Goods X and Y are jointly produced, with labor as the only input. The price of labor is one dollar per unit hired. In order to produce x units of X and y units of Y, the firm must hire $\max(x, y)$ units of labor. The market demand for the two goods is given by the functions $x = 2 - p_x$ and $y = 3 - p_y$.

- (a) Determine the competitive equilibrium production levels and prices of X and Y.
- (b) Determine the production levels and prices if the goods are produced by a single firm that has a monopoly in each of the two markets.

4.9 The only two goods in the economy are X and Y. There is only one firm in the economy, which can turn X into Y according to the production function $q = f(z)$, where $f(z) = 20\sqrt{z}$. Thus, z denotes the amount of X used as input, and Q denotes the amount of Y produced as output. There are two consumers, Amy and Bev, whose preferences are described by the utility functions

$$u_A(x_A, y_A) = 2x_A + y_A \quad \text{and} \quad u_B(x_B, y_B) = \frac{1}{2}x_B + y_B,$$

where x_i and y_i denote individual i 's consumption of X and Y. The economy has no endowment of Y, but each consumer owns 200 units of X. Amy owns the firm: she receives whatever profits it earns.

Determine all Walrasian equilibria. (Note that all consumers and firms are assumed to be price-takers in a Walrasian equilibrium, no matter how few of them there are.)

4.10 The tiny country of Dogpatch has 90 residents and 10 identical machines. Ten of the people (called “capitalists”) own one machine apiece, but are unable to provide any useful labor services. Each of the remaining 80 people (called “workers”) has the capacity to work with machines and the other workers to produce shmoos, but none of the workers owns any machines. Combining x workers with y machines yields $F(x, y)$ shmoos, where

$$F(x, y) = x^{\frac{2}{3}}y^{\frac{1}{3}} \quad \text{for all } (x, y) \in \mathbb{R}_+^2.$$

It is possible to divide a worker’s time among any number of machines and to divide a machine’s time among any number of workers.

No one in Dogpatch cares about consuming shmoos, but shmoos can be sold in the neighboring country of Alcappia for a dollar apiece. Everyone in Dogpatch uses dollars to purchase in Alcappia the goods that he does care about consuming. All residents of Dogpatch are price-takers in all markets, and everyone understands how to use the constant-returns-to-scale technology embodied in the function F . Machines and labor are neither imported nor exported by Dogpatch.

(a) Suppose that the capitalists are the entrepreneurs: each one hires workers and combines them with his machine, and then sells the resulting production of shmoos. What will be the equilibrium wage rate and total production of shmoos, and how many dollars will each capitalist and each worker spend in Alcappia?

(b) Now suppose the workers are the entrepreneurs: each worker rents machine time, which he combines with his own labor, and then sells the resulting production of shmoos. What will be the equilibrium rental price and total production of shmoos, and how many dollars will each capitalist and each worker spend in Alcappia?

(c) Is either of the institutional arrangements in (a) or (b) Pareto optimal for the residents of Dogpatch? If so, explain why; if not, find a Pareto improvement.

4.11 Alice owns no simoleons “today,” but she will own 30 simoleons “tomorrow.” Her preference for alternative consumption streams is described by the utility function $u_A(x, y) = \min\{x, y\}$, where x and y denote the number of simoleons she consumes today (x) and tomorrow (y). Betsy owns 20 simoleons today, but she will own none tomorrow; her preference is described by the utility function $u_B(x, y) = xy$.

(a) If Alice and Betsy engage in a borrowing-and-lending market (in which they’re the only participants) in order to arrive at more desirable consumption streams than they’re endowed with, and if each behaves “competitively” (taking the interest rate as given), what will be the equilibrium rate of interest and the equilibrium consumption stream of each?

(b) Verify that Walras’ Law is satisfied at the equilibrium rate of interest. Is Walras’ Law satisfied at any non-equilibrium interest rates, and if so, at which ones? Verify your answer.

(c) What is the net present value (NPV) of each woman’s wealth (i.e., of her endowment stream) at the equilibrium rate of interest?

(d) Now suppose the women’s endowment streams are (15,15) for Alice and (5,15) for Betsy. Is this situation Pareto optimal? Verify your answer.

(e) Is there an interest rate at which the endowment streams in (d) are a Walrasian equilibrium (i.e., an interest rate at which, if the women are endowed with intertemporal allocation ((15,15) (5,15)), there will be no intertemporal trade)? Are there any other endowment streams for which this allocation — i.e., (15,15) to Alice and (5,15) to Beth — is a Walrasian equilibrium? If so, determine a necessary condition on each woman’s wealth (the NPV of her endowment stream) that must be satisfied if the allocation ((15,15) (5,15)) is a Walrasian equilibrium.

For parts (f) and (g), assume that a single real investment process exists by which sacrificing simoleons today to be used as input will yield output of three times as many simoleons tomorrow.

(f) Assume that the women’s endowment streams are (20,0) for Alice and (10,0) for Betsy. Determine all Walrasian equilibrium interest rates, production plans, and consumption plans for each woman. What is the aggregate amount of profit? Does it matter who owns the investment (production) process? Why, or why not?

(g) Consider the plan in which 15 simoleons are used today as input to the investment process, Alice’s consumption stream is (5,5), and Betsy’s is (10,40). Find a Pareto improvement upon this plan (but not necessarily a Pareto optimal one) in which *both* women are strictly better off.

4.12 Jerry is shipwrecked on a tropical island. Fortunately, the island has a small grove of orange trees, and while Jerry doesn't care for oranges, he does like orange juice. Even *more* fortunately, he saved two orange juice machines before his ship went down. Jerry refers to the machines as Firm 1 and Firm 2. Each machine is capable of producing orange juice from oranges, but one machine is more efficient than the other. Specifically, the machines turn z oranges per day into q ounces of orange juice per day according to the production functions

$$q_1 = f_1(z_1) = 12(\sqrt{z_1 + 1} - 1) \quad \text{and} \quad q_2 = f_2(z_2) = 12(\sqrt{z_2 + 4} - 2).$$

Note that the machines' marginal productivities are

$$f'_1(z_1) = 6/\sqrt{z_1 + 1} \quad \text{and} \quad f'_2(z_2) = 6/\sqrt{z_2 + 4}.$$

Firm 1 is uniformly more efficient than Firm 2: for any input z of oranges, $f_1(z) > f_2(z)$; moreover, Firm 1's *marginal* productivity is also larger at any level of operation: $f'_1(z) > f'_2(z)$. Should Jerry therefore use Firm 1 exclusively? Let's find out.

(a) Suppose Jerry's orange grove yields 13 oranges each day. He simply wants to use the 13 oranges as inputs into his machines every day in whatever way will produce the most orange juice. Determine the greatest amount of orange juice Jerry can produce each day, and determine how many oranges he must put into each machine each day in order to obtain that maximum level of production.

(b) Instead of 13 oranges per day, suppose Jerry's orange grove yields only 3 oranges per day. How many oranges should he put into each machine and how much orange juice will he produce? What if his orange grove yields *fewer* than 3 oranges per day?

Jerry decides that he likes oranges after all. His preferences for oranges and orange juice are described by the utility function

$$u(x, y) = x + 2y - \frac{1}{48}y^2,$$

where x and y denote, respectively, oranges consumed per day and ounces of orange juice consumed per day. Jerry's orange grove is now producing 30 oranges per day.

(c) How many oranges per day will Jerry consume and how many will he put into each machine? How much orange juice will he produce and consume? Suppose he wants to decentralize this plan; what are the decentralizing (*i.e.*, efficiency) prices he will need to use? How much profit should he attribute to each "firm?"

Jerry discovers a second survivor of the shipwreck — Kramer! The two of them agree to divide up ownership of the island’s resources. Kramer assumes 100% ownership of the eastern 1/3 of the orange grove (yielding 10 oranges per day) and 100% ownership of the more efficient firm, Firm 1. Jerry assumes 100% ownership of the less efficient Firm 2 and, to compensate for getting the less efficient firm, he receives 100% ownership of the western 2/3 of the orange grove (yielding 20 oranges per day). Jerry’s and Kramer’s preferences are described by the utility functions

$$u_J(x, y) = x + 2y - \frac{1}{40}y^2 \quad \text{and} \quad u_K(x, y) = x + y - \frac{1}{24}y^2$$

(d) If Jerry and Kramer each behave as price-takers in their consumption and production decisions, and each one operates his firm so as to maximize its profit, what will be the equilibrium prices? How much will each one consume of each good, how many oranges will each firm purchase and use as inputs, how much orange juice will each firm produce, and how much profit will each firm earn? Will the outcome be Pareto efficient?

(e) How would your answers to (d) change if ownership of the firms were different? For example, what if ownership were reversed, Jerry owning Firm 1 and Kramer Firm 2? What if each one owns 50% of each firm?

(f) Determine Jerry’s and Kramer’s consumer surplus and the firms’ producer surplus. Is the total surplus a good measure of an outcome’s welfare?

4.13 As in Harberger’s classic example [JPE 1962], assume that there are two goods produced: product X is produced by firms in the “corporate” sector and product Y by firms in the “non-corporate” sector. Both products are produced using the two inputs **labor** and **capital** (quantities denoted by L and K). Production functions are

$$X = \sqrt{L_X K_X} \quad \text{and} \quad Y = \sqrt{L_Y K_Y}.$$

All consumers have preferences described by the utility function $u(X, Y) = XY$. The consumers care only about consuming X and Y , and they supply labor and capital inelastically in the total amounts $L = 600$ and $K = 600$. Let p_X , p_Y , p_L , and p_K denote the prices of the four goods in the economy; assume that $p_L = 1$ always.

(a) What is the Walrasian equilibrium?

(b) Suppose that a 50% tax is imposed on payments to capital in the corporate sector only, and that the government uses the tax proceeds to purchase equal amounts of the output of the two sectors. What will be the new Walrasian equilibrium? How is welfare affected by the tax – are people better off with or without the tax?

4.14 Benjamin has just graduated from college. Now he's deciding how much further to invest in his own human capital. The result of his decision will be a consumption stream $\mathbf{x} = (x_0, x_1)$, where x_0 denotes consumption "today" (say, during the next ten years) and x_1 denotes consumption "tomorrow" (the remainder of his life). His preference is described by the utility function $u(x_0, x_1) = x_0^3 x_1^2$. If he undertakes no investment, and neither saves nor borrows, his consumption stream will be $\hat{\mathbf{x}} = (\hat{x}_0, \hat{x}_1) = (38, 16)$.

The investment possibilities available to Benjamin are described by the function

$$f(z) = \begin{cases} 4z - \frac{1}{8}z^2, & z \leq 16 \\ 32, & z \geq 16. \end{cases}$$

(a) Suppose Benjamin has no access to capital markets: he can neither borrow nor save. Verify that he will invest at level $z = 8$. What will be his resulting consumption stream, marginal rate of substitution, and marginal rate of transformation? Depict this decision in a diagram showing Benjamin's consumption-possibilities set and the best indifference curve he can attain.

Now suppose Benjamin has access to capital markets in which he can borrow and lend (i.e., save) at a common inter-period (not annual) interest rate, r – i.e., if he borrows B dollars today, he repays $(1 + r)B$ dollars tomorrow; if he saves S today, he receives $(1 + r)S$ tomorrow.

(b) If the interest rate is 100%, how much will Benjamin invest? How much will he borrow or lend? What will be his resulting consumption stream? Verify that the net present value of his consumption stream is equal to the net present value of his endowment stream plus the net present value of his investment plan.

(c) Now suppose the interest rate is 200%, and consider the investment and consumption plans in part (b). Verify that the present value of the consumption stream in (b) is still equal to the present value of Benjamin's endowment stream plus the present value of the investment plan in (b). In a diagram like the one in part (a), depict the present-value contour(s) on which the investment plan and consumption plan lie. Will Benjamin's investment and consumption plans be the same as in (b)? If not, will his investment, borrowing, lending, and consumption in each period be more or less than in (b)? Hint: If $r = 200\%$, the present value of the investment plan in part (b) is zero.

(d) If the interest rate is 200%, how much will Benjamin invest? How much will he borrow or lend? What will be his resulting consumption stream? Hint: Benjamin will choose a consumption plan that maximizes his utility among all plans with a net present value no greater than his wealth. His wealth is the present value of his endowment plus the net present value of his investment plan. You should find that his wealth is 44.