Pareto Improvements and Pareto Efficiency

3.1 Assume throughout this exercise that

P is an irreflexive relation on a set X

P^c denotes the complement of P — *i.e.*, xP^cy if and only if "not xPy"

$I := P^c \cup (P^{-1})^c$ — *i.e.*, xIy if and only if neither xPy nor yPx

$R := P \cup I$ — *i.e.*, xRy if and only if xPy or xIy

$N = \{1, \ldots, n\}$.

For any list $(P_1, ..., P_n)$ of preference relations, let $\overline{P}$ denote the associated Pareto relation, *i.e.*, the Pareto aggregation of $(P_1, ..., P_n)$: $x\overline{P}y$ if $\left[\exists i \in N : xP_iy \text{ and } \nexists i \in N : yP_ix\right]$.

(a) Prove that if P_i is irreflexive for each $i \in N$, then $\overline{P}$ is irreflexive.

(b) Prove that the relation R is transitive if and only if its associated P and I are both transitive.

(c) Provide a counterexample to the following proposition: "If, for each $i \in N$, P_i is transitive, then $\overline{P}$ is transitive." (Try to find the simplest possible counterexample. It might help to use the interpretation that the elements of X are universities, or economics departments, or basketball teams, etc. It may also help in this case to remember that a binary relation on a set X is a subset of $X \times X$.)

(d) Prove that if, for an irreflexive relation P, the associated R is transitive, then

(i) xPy & $yRz \Rightarrow xPz$

(ii) xRy & $yPz \Rightarrow xPz$.

(e) Prove that if R_i is transitive for each $i \in N$, then $\overline{P}$ is transitive.

The lecture notes provide examples which show that if each R_i is transitive, $\overline{I}$ need not be transitive, and thus, according to (b) above, $\overline{R}$ need not be transitive.

3.2 Assume that each consumer's utility function on $\mathbb{R}^2_+$ is continuously differentiable, quasiconcave, and strictly increasing.

(a) Show diagrammatically that the following statement is true for any bundle $(x, y) \in \mathbb{R}^2_+$:

($*$) A change $(\Delta x, \Delta y)$ in the bundle will make the consumer worse off if $\Delta y < (MRS)(-\Delta x)$.

(b) In Exercise 1.6 determine all interior Pareto allocations and depict them in an Edgeworth box diagram.

(c) Someone has proposed that the endowment $(\mathring{x}, \mathring{y}) = (60, 30)$ of apples and oranges from the two families' orchards in Exercise 1.6 be allocated as follows: the Arrow family would receive the bundle $(\widehat{x}_A, \widehat{y}_A) = (20, 30)$ and the Debreu family would receive the bundle $(\widehat{x}_D, \widehat{y}_D) = (40, 0)$. In the Edgeworth box, depict each household's indifference curve through the proposed allocation. Use the definition of Pareto efficiency and the condition ($*$) above to verify that this proposal is Pareto efficient in spite of the fact that $MRS_A < MRS_D$ at the proposed allocation.

(d) In Exercise 1.4 determine all interior Pareto allocations and depict them in an Edgeworth box diagram. Does the argument in (c) work here for the allocation in which $(x_A, y_A) = (2, 3)$ and $(x_B, y_B) = (6, 0)$?

3.3 There are only two goods and two consumers in the economy, and no production is possible. The consumers' preferences can be represented by the utility functions

$$u^1(x, y) = y + \log(1 + x) \quad \text{and} \quad u^2(x, y) = y + 2\log(1 + x).$$

for all bundles in which $x, y \geqq 0$. Each consumer is endowed with 5 units of each good. Determine all interior Pareto allocations and depict them in an Edgeworth box diagram. Consider all the allocations in which $y_A = 10$ and $y_B = 0$; to which of these allocations does the "boundary" argument in Exercise 3.2 apply? Can you make a similar argument about any of the allocations in which $y_A = 0$ and $y_B = 10$?

3.4 (See Exercise 1.5) There are two consumers, Al and Bill, and two goods, the quantities of which are denoted by x and y. Al and Bill each own 100 units of the Y-good; Al owns 12 units of the X-good and Bill owns 3 units. Their preferences are described by the utility functions

$$u_A(x_A, y_A) = y_A + 60x_A - 2x_A^2 \quad \text{and} \quad u_B(x_B, y_B) = y_B + 30x_B - x_B^2.$$

Note that their marginal rates of substitution are $MRS_A = 60 - 4x_A$ and $MRS_B = 30 - 2x_B$.

Determine the entire set of Pareto allocations. (You may do this via MRS conditions.) Depict the set in an Edgeworth box diagram. (Use different scales on the x- and y-axes or your diagram will be very tall and skinny.)

3.5 Ann and Bill work together as water ski instructors in Florida. Each earns \$100 per day. Each one also owns orange trees that yield 8 oranges per day. Ann likes oranges "more" than Bill does; specifically, Ann's MRS for oranges is $MRS_A = 12 - x_A$ and Bill's MRS is $MRS_B = 8 - x_B$, where x_i denotes i's daily consumption of oranges and the MRS tells how many dollars (i.e., how much consumption of other goods) one would be willing to give up to get an additional orange.

(a) Bill has been selling two oranges a day to Ann, for which Ann has been paying Bill \$3 per day. (Thus, Ann ends up with 10 oranges and \$97 per day, and Bill ends up with 6 oranges and \$103 per day.) Is this Pareto efficient? Are they both better off than they would be if they did not trade? Is this a Walrasian Equilibrium? Verify your answers.

(b) In an Edgeworth box diagram depict clearly *all* Pareto efficient allocations of oranges and dollars to Ann and Bill.

(c) A hurricane has destroyed Ann's orange crop but has left Bill's crop undamaged. The Florida legislature has hurriedly passed a law against "price gouging." The law specifies that oranges cannot be sold for more than four dollars apiece. At the price of four dollars, Bill is willing to sell Ann four oranges per day, but not more. Would Ann be willing to buy four oranges at four dollars apiece? Are there illegal trades (i.e., at a price of more than four dollars per orange) that would make them both better off than they are at the legal trade of four oranges for four dollars apiece? If so, find such a trade; if not, explain why not.

(d) Determine whether the Walrasian equilibrium (after the hurricane) is a Pareto improvement on the allocation in (c), in which Bill sells Ann four oranges per day for four dollars apiece.

3.6 (See Exercise 1.2) Ann and Bob each own 10 bottles of beer. Ann owns 20 bags of peanuts and Bob owns no peanuts. There are no other people and no other goods in the economy, and no production of either good is possible. Using x to denote bottles of beer and y to denote bags of peanuts, Ann's and Bob's preferences are described by the following utility functions:

$$u_A(x_A, y_A) = x_A y_A^4 \qquad \text{and} \qquad u_B(x_B, y_B) = 2x_B + y_B.$$

Note that their MRS schedules are $MRS_A = y_A/4x_A$ and $MRS_B = 2$.

(a) Determine all Walrasian equilibrium price lists and allocations.

(b) Determine all boundary allocations that are Pareto efficient.

(c) Determine all interior allocations that are Pareto efficient, and draw the set of all Pareto efficient allocations in an Edgeworth box.

3.7 There are two goods (quantities x and y) and two people (Ann and Bob) in the economy. Ann owns two units of each good and Bob owns six units of each good. Their preferences are described by the utility functions:

$$u^A(x_A, y_A) = x_A^2 y_A \quad \text{and} \quad u^B(x_B, y_B) = y_B - \frac{1}{2}(8 - x_B)^2.$$

(a) Derive the complete marginal conditions that characterize the Pareto optimal allocations (*i.e.*, the complete first-order marginal conditions for an allocation to be a solution of the problem (P-Max)), and use these conditions to determine the set of all Pareto allocations. Draw this set in an Edgeworth box diagram.

(b) Determine the competitive equilibrium price(s) and allocation(s).

(c) For each of the following allocations determine whether the allocation is Pareto optimal. If it is, give all the "decentralizing" price lists, and determine all the *initial* allocations for which the given allocation is a Walrasian equilibrium. If the given allocation is not Pareto optimal, verify that there are no values of the Lagrange multipliers for which the given allocation satisfies the first-order conditions in (a) above, and find a Pareto optimal allocation that makes Ann and Bob both strictly better off.

(c1)	(x_A, y_A)	$=$	(6,8),	(x_B, y_B)	$=$	(2,0)
(c2)	(x_A, y_A)	$=$	(8,2),	(x_B, y_B)	$=$	(0,6)
(c3)	(x_A, y_A)	$=$	(4,8),	(x_B, y_B)	$=$	(4,0)
(c4)	(x_A, y_A)	$=$	$(3,4\frac{1}{2})$,	(x_B, y_B)	$=$	$(5,3\frac{1}{2})$

3.8 There are two goods (quantities x and y) and two people (Andy and Bea) in the economy. No production is possible. An allocation is a list (x_A, y_A, x_B, y_B) specifying what each person receives of each good. Andy's and Bea's preferences are described by the utility functions

$$u^A(x_A, y_A) = 2x_A + y_A + \alpha \log x_B \qquad \text{and} \qquad u^B(x_B, y_B) = x_B + y_B.$$

The two goods are available in the positive amounts $\mathring{x}$ and $\mathring{y}$, and α satisfies $0 < \alpha < \mathring{x}$. Note that Andy cares directly about how much Bea receives of the x-good.

Determine all the Pareto efficient allocations in which Andy and Bea both receive a positive amount of each good.

3.9 (See Exercise 1.8) There are r girls and r boys, where r is a positive integer. The only two goods are bread and honey, quantities of which will be denoted by x and y: x denotes loaves of bread and y denotes pints of honey. Neither the girls nor the boys are well endowed: each girl has 8 pints of honey but no bread, and each boy has 8 loaves of bread but no honey. Each girl's preference is described by the utility function $u_G(x, y) = \min(ax, y)$ and each boy's by the utility function $u_B(x, y) = x + y$.

(a) Determine the Walrasian excess demand function for honey and the Walrasian equilibrium prices and allocations.

(b) Determine the set of Pareto optimal allocations for $r = 1$ and for arbitrary r.

(c) Assume that $a = 1$. Determine the core allocations for $r = 1$, for $r = 2$, and for arbitrary r.

3.10 Amy owns five bottles of wine, but no cheese. Bob owns ten pounds of cheese, but no wine. Their preferences for wine and cheese are described by the following marginal rates of substitution (x denotes wine consumption, in bottles, and y denotes cheese consumption, in pounds):

$$\text{Amy:} \quad MRS_A = \begin{cases} 5, & \text{if } x < 3 \\ 1, & \text{if } x > 3 \end{cases} \qquad\qquad \text{Bob:} \quad MRS_B = 6 - x.$$

(a) Draw Amy's indifference curve that contains the bundle (3,3). Is Amy's preference representable by a continuous utility function? If so, give such a function; if not, indicate why not. Draw Bob's indifference curve through the bundle (4,2).

(b) In an Edgeworth box diagram, depict the entire set of Pareto optimal allocations

(c) Determine all Walrasian equilibrium price lists and allocations.

(d) Suppose Amy and Bob are joined by Ann and Bill. Ann is exactly like Amy (same preferences, same endowment), and Bill is exactly like Bob. So, now there are two people of each type. Show that the following allocation is not in the core: Amy and Ann each get (3,2), and Bob and Bill each get (2,8).

3.11 There are two goods (quantities x and y) and two people (Amy and Bev) in the economy. No production is possible. There are 30 units of the x-good and 60 units of the y-good available to be distributed to Amy and Bev, whose preferences are as follows:

> Amy's MRS is 3 if $y > x$ and her MRS is $1/2$ if $y < x$;
> Bev's MRS is always 1.

(a) Draw an Edgeworth box diagram and indicate on the diagram the entire set of Pareto optimal allocations.

(b) If Amy owns the bundle (20,60) and Bev owns the bundle (10,0), determine the competitive (Walrasian) equilibrium price(s) and allocation(s).

3.12 (See Exercise 1.9) There are only two consumers, Amy and Bev, and only two goods, the quantities of which are denoted by x and y. There are 20 units of each good to be allocated between Amy and Bev. Amy's and Bev's preferences can be represented by the utility functions

$$u_A(x_A, y_A) = \log x_A + 4 \log y_A \quad \text{and} \quad u_B(x_B, y_B) = y_B + 5 \log x_B.$$

(a) Determine the set of all Pareto allocations and depict the set carefully in an Edgeworth box diagram. (You may do this via MRS conditions.)

(b) Verify that the allocation $((x_A, y_A), (x_B, y_B)) = ((4, 5), (16, 15))$ is Pareto efficient by finding values of the Lagrange multipliers in the first-order conditions for the problem (P-max) and then showing that with these Lagrange values the first-order conditions are indeed satisfied.

(c) Now assume that Amy owns the bundle $(4, 5)$ and Bev owns the bundle $(16, 15)$. Determine a Walrasian equilibrium, and verify by direct appeal to the definition that the equilibrium you have identified is indeed an equilibrium.

(d) Verify that the allocation $((x_A, y_A), (x_B, y_B)) = ((12, 20), (8, 0))$ is Pareto efficient by finding values of the Lagrange multipliers in the first-order conditions for the problem (P-max) and then showing that with these Lagrange values the first-order conditions are indeed satisfied.

3.13 (See Exercise 1.4) There are two goods (quantities x and y) and two people (Al and Bill) in the economy. Al owns eight units of the x-good and none of the y-good. Bill owns none of the x-good, and three units of the y-good. Their preferences are described by the utility functions

$$u^A(x_A, y_A) = x_A y_A \quad \text{and} \quad u^B(x_B, y_B) = y_B + \log x_B.$$

(a) Determine the competitive equilibrium price(s) and allocation(s).

(b) Derive the complete marginal conditions that characterize the Pareto optimal allocations, and draw the set of all Pareto optimal allocations in an Edgeworth box diagram.

(c) For each of the following allocations determine whether the allocation is Pareto optimal. If it is, give all the "decentralizing" price lists; if it isn't, find a Pareto optimal allocation that makes both Al and Bill strictly better off.

$$\begin{array}{llllll}
\text{(c1)} & (x_A, y_A) & = & (4{,}1), & (x_B, y_B) & = & (4{,}2) \\
\text{(c2)} & (x_A, y_A) & = & (1{,}3), & (x_B, y_B) & = & (7{,}0) \\
\text{(c3)} & (x_A, y_A) & = & (4{,}2), & (x_B, y_B) & = & (2{,}1) \\
\text{(c4)} & (x_A, y_A) & = & (7{,}3), & (x_B, y_B) & = & (1{,}0)
\end{array}$$

3.14 (See Exercise 1.8) Quantities of the economy's only two goods are denoted by x and y; no production is possible. Ann's and Ben's preferences are described by the utility functions

$$u^A(x, y) = ax + y \qquad \text{and} \qquad u^B(x, y) = x^b y.$$

(a) Let w_x and w_y denote the available amounts of the two goods. Determine all the Pareto efficient allocations, expressing them in terms of the parameters a, b, w_x, and w_y. For each of following three cases, draw an Edgeworth box diagram and indicate on the diagram the entire set of Pareto efficient allocations:

$$\text{Case I: } \frac{w_y}{w_x} = \frac{a}{b} \qquad \text{Case II: } \frac{w_y}{w_x} > \frac{a}{b} \qquad \text{Case III: } \frac{w_y}{w_x} < \frac{a}{b}.$$

(b) Let $a = b = 1$, and suppose that Ann owns the bundle (0,5) and Ben owns the bundle (30,5). Determine the Walrasian equilibrium price(s) and allocation(s).

3.15 There are two goods (quantities x and y) in the economy and two people, Alex and Beth, whose preferences are described by the utility functions

$$u^A(x_A, y_A) = x_A + 2y_A \qquad \text{and} \qquad u^B(x_B, y_B) = y_B - \frac{1}{2}(12 - x_B)^2.$$

Let $\mathring{x}_i$ and $\mathring{y}_i$ denote i's initial holdings ($i = A, B$), and assume that between them Alex and Beth own a total of 10 units of each good. Let r denote the ratio $\mathring{y}_B/\mathring{x}_A$, and consider the following three cases:

$$\text{Case I: } r > 2 \qquad \text{Case II: } \tfrac{1}{2} < r < 2 \qquad \text{Case III: } r < \tfrac{1}{2}.$$

(a) Assuming we're in Case I, determine the complete first-order conditions that characterize the Pareto optimal allocations in terms of marginal rates of substitution. Draw the set of all Pareto optimal allocations in an Edgeworth box diagram.

(b) Describe informally how the set of Pareto optimal allocations and first-order conditions in (a) are changed if we're in Case II or Case III.

(c) Assuming that each person owns five units of each good before trading, determine all the competitive equilibrium price(s) and allocation(s).

(d) Assuming that Beth owns all ten units of the y-good, and that each person owns fives units of the x-good, determine all the competitive equilibrium price(s) and allocation(s).

(e) Determine whether it is Pareto optimal for Alex to be given all of the y-good and Beth all of the x-good. If so, determine all the decentralizing prices; if not, find a Pareto improvement.

(f) Determine the competitive equilibrium prices in Case I, Case II, and Case III.

3.16 (See Exercise 1.6) Amy and Bob consume only two goods, quantities of which we'll denote by x and y. Amy and Bob have the same preferences, described by the utility function

$$u(x,y) = \begin{cases} x + y - 1, & \text{if } x \geqq 1 \\ 3x + y - 3, & \text{if } x \leqq 1. \end{cases}$$

There are 4 units of the x-good, all owned by Amy, and 6 units of the y-good, all owned by Bob.

(a) Draw the Edgeworth box diagram, including each person's indifference curve through the initial endowment point. Determine all Walrasian equilibrium prices and allocations.

(b) In an Edgeworth box diagram, depict all Pareto optimal allocations.

(c) In an Edgeworth box diagram, depict all core allocations.

Suppose Cal joins Amy and Bob. Cal owns 18 units of the y-good but none of the x-good, and he has preferences described by the utility function $u(x,y) = 2x + y$.

(d) Determine all competitive equilibrium prices and allocations.

(e) Show that now, with Cal present, none of the core allocations give Amy and Bob what they received in any of the competitive allocations in (a).

3.17 The economy consists of two people (Mr. A and Mr. B) and two goods (the quantities of which will be denoted by x and y). Mr. A owns all the x-good (4 units) and Mr. B owns all the y-good (6 units). It is not possible to produce any additional units of either good. Let (x_i, y_i) denote the bundle allocated to (or consumed by) Mr. i, where i may be either A or B. The two people's preferences are described by the following utility functions

$$\begin{aligned} u_A(x_A, y_A) &= \begin{cases} y_A + 3x_A, & \text{if } x \leqq 2 \\ y_A + \frac{1}{2}x_A + 5, & \text{if } x \geqq 2 \end{cases} \\ u_B(x_B, y_B) &= y_B - \tfrac{1}{2}(4 - x_B)^2. \end{aligned}$$

(a) Depict the set of all Pareto optimal allocations in an Edgeworth box diagram.

(b) Determine all the Walrasian equilibrium price lists and allocations, and depict them in an Edgeworth box diagram.

(c) Suppose that another person just like Mr. A (same preferences, same endowment) is added to the economy, and also another person just like Mr. B. (So now there are two people of each type.) Show that the following allocation is not in the core: each type-A person gets $(2, 1)$ and each type-B person gets $(2, 5)$.

3.18 The following theorem appears in the lecture notes: If every u^i is continuous and locally nonsatiated, then an interior allocation $\hat{\mathbf{x}}$ is Pareto efficient for the economy $(u^i, \mathring{\mathbf{x}}^i)_1^n$ if and only if it is a solution of the problem (P-Max).

(a) Provide a counterexample to show why, for interior allocations, the theorem requires that utility functions be locally nonsatiated.

(b) Provide a counterexample to show why, at a boundary allocation, local nonsatiation is not enough — a boundary allocation could be a solution of (P-Max) but not Pareto efficient, even if every u^i is continuous and locally nonsatiated.

3.19 There are two goods (quantities are denoted by x and y) and two people (Alex and Beth), whose preferences are described by the utility functions

$$u^A(x,y) = xy \qquad \text{and} \qquad u^B(x,y) = 2x + y.$$

There are eight units of the x-good to be allocated and six units of the y-good. Someone has proposed that the bundles $(x_A, y_A) = (2,4)$ and $(x_B, y_B) = (6,2)$ be allocated to Alex and Beth.

(a) Determine the gradients ∇u^A and ∇u^B at the proposal. Draw Alex's and Beth's consumption spaces, including the bundles they would receive in the proposal, their indifference curves through those bundles, and the gradients at those bundles. Is $\nabla u^A = \lambda \nabla u^B$ for some λ?

(b) Write down the Pareto maximization problem (P-Max), obtain the first-order marginal conditions (FOMC), and then evaluate the first-order conditions at the proposal. (Use the notation σ_x and σ_y for the Lagrange multipliers associated with the feasibility constraints, and λ for the Lagrange multiplier associated with the constraint on Beth's utility level.) Determine whether the proposal satisfies the FOMC — *i.e.*, determine whether there are values of the three Lagrange multipliers for which the FOMC are satisfied at the proposal.

(c) Determine whether each gradient ∇u^i is a multiple of the vector (σ_x, σ_y).

(d) Determine Alex's and Beth's marginal rates of substitution at the proposal.

(e) Someone else has proposed that the bundles $(x_A, y_A) = (6,2)$ and $(x_B, y_B) = (2,4)$ be allocated to Alex and Beth, the reverse of the first proposal. Answer the same questions (a)-(d) for this second proposal.

(f) Determine all the interior Pareto allocations to Alex and Beth. Draw the set of these allocations in an Edgeworth box diagram.

(g) Consider a third proposal, $(x_A, y_A) = (6,6)$ and $(x_B, y_B) = (2,0)$. Determine whether the FOMC for the problem (P-Max) are satisfied for this proposal.

3.20 One possible social welfare criterion for choosing among alternative allocations is the sum of individuals' utilities, or a weighted sum of the utilities:

$$W(\mathbf{x^1}, \ldots, \mathbf{x}^n) = \sum_{i=1}^{n} \theta_i u^i(\mathbf{x}^i),$$

where $\theta_1, \ldots, \theta_n$ are exogenously given weights (positive real numbers).

Assume there are just two goods and two consumers, with utility functions of the form

$$u_i(x, y) = \alpha_i \log x + \beta_i \log y.$$

Assume that $\mathring{x}$ and $\mathring{y}$ are the total amounts of the goods that are available to distribute to the two consumers.

(a) Determine the allocation(s) that maximize $W(\cdot)$ as a function of the eight parameters $\theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \mathring{x}$, and $\mathring{y}$.

Solution:

$$x_i = \frac{\theta_i \alpha_i}{\theta_1 \alpha_1 + \theta_2 \alpha_2}\, \mathring{x} \qquad \text{and} \qquad y_i = \frac{\theta_i \beta_i}{\theta_1 \beta_1 + \theta_2 \beta_2}\, \mathring{y}\,, \qquad i = 1, 2\,.$$

(b) Determine which, if any, of the allocations that maximize $W(\cdot)$ also satisfy the condition $MRS_1 = MRS_2$.

(c) Assume that $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$ and $(\mathring{x}, \mathring{y}) = (30, 60)$. What is the "welfare maximizing" allocation if $\theta_1 = \theta_2$? Depict this situation in an Edgeworth box diagram. As the θ's vary over all possible values, determine the set of allocations that could possibly maximize welfare for *some* value(s) of θ.

(d) Assume that $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 2$, and $(\mathring{x}, \mathring{y}) = (30, 60)$. What is the "welfare maximizing" allocation if $\theta_1 = \theta_2$? Depict this situation in an Edgeworth box diagram. As the θ's vary over all possible values, determine the set of allocations that could possibly maximize welfare for *some* value(s) of θ.

(e) Compare the allocation in (c) to the allocation in (d) for arbitrary values of θ_1 and θ_2.

(f) How do the consumers' indifference maps in (d) differ from their maps in (c)?

3.21 In Exercise #1.15 identify all the Pareto allocations.