EXERCISE SOLUTION

BECAUSE $u^1(\mathring{x}^1) = u^2(\mathring{x}^2) = 19$ AND $u^3(\mathring{x}^3) = 100$, EVERY CORE ALLOCATION MUST SATISFY $u^1, u^2 \geqq 19$ AND $u^3 \geqq 100$. IN PARTICULAR, EACH x_k^i MUST BE STRICTLY POSITIVE. AND SINCE EACH CORE ALLOCATION MUST BE PARETO OPTIMAL, WE MUST THEREFORE HAVE $MRS^1 = MRS^2 = MRS^3$ — i.e., $x_1^i = x_2^i$ FOR EACH i, SINCE $\mathring{x}^1 + \mathring{x}^2 + \mathring{x}^3 = (30,30)$. LET z_i DENOTE THE AMOUNT OF EACH GOOD ALLOCATED TO i IN A CORE ALLOCATION — i.e., $x_1^i = x_2^i = z_i$; WE WILL DESCRIBE THE CORE ALLOCATIONS VIA RESTRICTIONS ON z_1, z_2, z_3.

THE RESTRICTION IMPLIED BY PARETO OPTIMALITY — i.e., BY ENSURING THAT $N = \{1,2,3\}$ CANNOT IMPROVE UPON (z_1, z_2, z_3) — IS $z_1 + z_2 + z_3 = 30$.

INDIVIDUAL RATIONALITY — i.e., THAT NONE OF THE COALITIONS $\{1\}$, $\{2\}$, OR $\{3\}$ CAN IMPROVE — YIELDS $z_1 \geqq \sqrt{19}$, $z_2 \geqq \sqrt{19}$, AND $z_3 \geqq \sqrt{100} = 10$.

THE COALITION $\{1,2\}$ OWNS THE BUNDLE $(20,20)$; THEIR UTILITY FRONTIER IS THEREFORE GIVEN BY $\sqrt{u_1} + \sqrt{u_2} = 20$, AND THEREFORE WE MUST HAVE $z_1 + z_2 \geqq 20$.

THE COALITION $\{1,3\}$ OWNS THE BUNDLE $(29,11)$; THEIR UTILITY FRONTIER IS THEREFORE GIVEN BY $\sqrt{u_1} + \sqrt{u_3} = \sqrt{319}$, AND THEREFORE WE MUST HAVE $z_1 + z_3 \geqq \sqrt{319}$. SIMILARLY, $z_2 + z_3 \geqq \sqrt{319} \cong 17.86$.

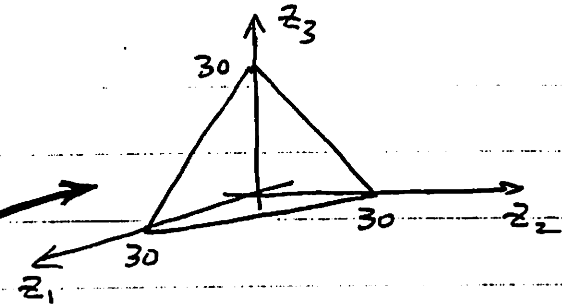
COMBINING ALL THESE INEQUALITIES YIELDS $z_3 \geqq 10$, $z_1 + z_2 \geqq 20$, AND $\sqrt{19} \leqq z_1 \leqq \sqrt{319}$, $\sqrt{19} \leqq z_2 \leqq \sqrt{319}$.

$S = \{1,2,3\}$

$\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} \geqq 30$

i.e., $z_1 + z_2 + z_3 \geqq 30$

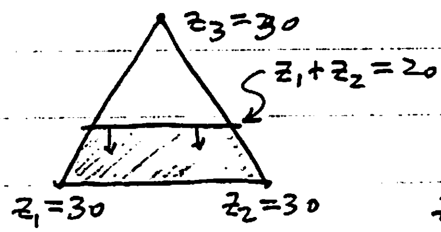
$\therefore z_1 + z_2 + z_3 = 30$



$S = \{1,2\}$

$\sqrt{u_1} + \sqrt{u_2} \geqq 20$

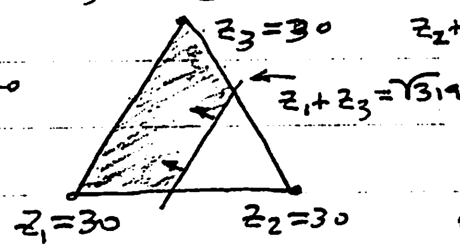
i.e., $z_1 + z_2 \geqq 20$



$S = \{1,3\}$

$\sqrt{u_1} + \sqrt{u_3} \geqq \sqrt{(29)(11)}$

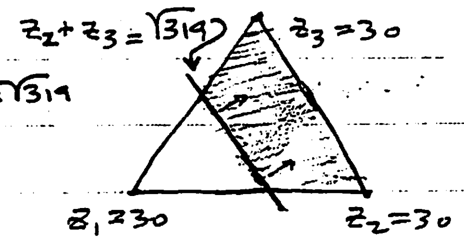
i.e., $z_1 + z_3 \geqq \sqrt{319}$



$S = \{2,3\}$

$\sqrt{u_2} + \sqrt{u_3} \geqq \sqrt{(11)(29)}$

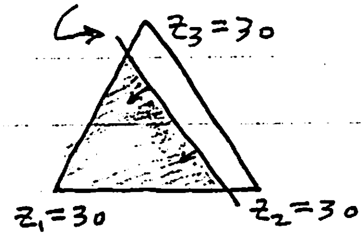
i.e., $z_2 + z_3 \geqq \sqrt{319}$



$S = \{1\}$

$u_1 \geqq 19$

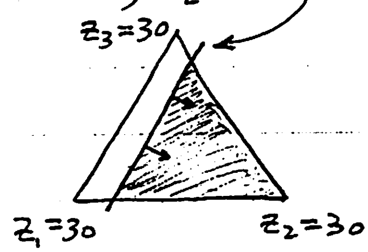
i.e., $z_1 \geqq \sqrt{19}$



$S = \{2\}$

$u_2 \geqq 19$

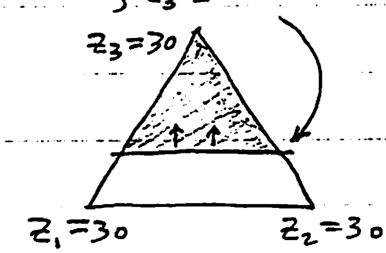
i.e., $z_2 \geqq \sqrt{19}$



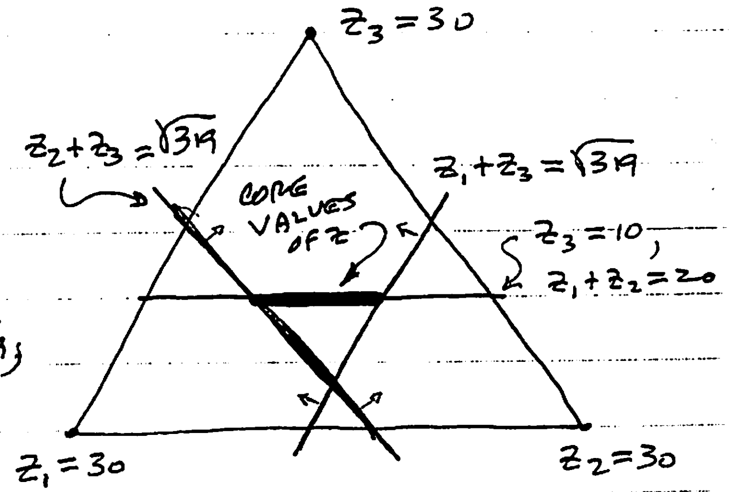
$S = \{3\}$

$u_3 \geqq 100$

i.e., $z_3 \geqq \sqrt{100} = 10$



THE CORE ALLOCATIONS CORRESPOND TO THE POINTS $z = (z_1, z_2, z_3)$ THAT SATISFY ALL THE ABOVE CONSTRAINTS: $z_3 = 10$; $z_1 + z_2 = 20$; $z_2 + z_3 \geqq \sqrt{319}$; $z_1 + z_3 \geqq \sqrt{319}$. [$z_1 \geqq \sqrt{19}$ AND $z_2 \geqq \sqrt{19}$ ARE NOT BINDING.]



AS THE VALUE FUNCTION FOR THE PROBLEM (P-MAX)

(FOR $S=\{1,2,3\}$)

DERIVING THE UTILITY FRONTIER:

$$\max \; u^3(x^3) \quad \text{s.t.} \quad x^i \geqq \underline{0}, \; \forall i$$

AND TO

$$u^1(x^1) \geqq u^1$$
$$u^2(x^2) \geqq u^2$$
$$\sum x_1^i \leqq 30$$
$$\sum x_2^i \leqq 30$$

FIRST-ORDER CONDITIONS YIELD:

(IN INTERIOR)

$$x_1^i = x_2^i, \; \forall i \qquad (= z_i, \text{ SAY})$$

$$\therefore \; z_1 = \sqrt{u^1}; \quad z_2 = \sqrt{u^2};$$

$$\text{i.e., } x_1^1 = x_2^1 = \sqrt{u^1}, \quad x_1^2 = x_2^2 = \sqrt{u^2};$$

$$\therefore \; x_1^3 = x_2^3 = 30 - (\sqrt{u^1} + \sqrt{u^2})$$

AND $\boxed{u^3 = \left[30 - (\sqrt{u^1} + \sqrt{u^2})\right]^2}$

$$\text{i.e., } \sqrt{u^3} = 30 - (\sqrt{u^1} + \sqrt{u^2})$$

$$\text{i.e., } \boxed{\sqrt{u^1} + \sqrt{u^2} + \sqrt{u^3} = 30}$$