Econ 501B Fall 2006
First Midterm Exam

Solutions

1. \( u_A(x, y) = \log x + 4 \log y \), \( u_B(x, y) = y + 5 \log x \)

\( u_A_x = \frac{1}{x}, \quad u_A_y = \frac{4}{y} \quad u_B_x = \frac{5}{x}, \quad u_B_y = 1 \)

\( MRS_A = \frac{4}{x} \cdot \frac{y}{x} \)

\( MRS_B = \frac{5}{y} \)

2. \( MRS_A = MRS_B = \frac{\frac{1}{4} \cdot \frac{y}{x}}{\frac{1}{20} - \frac{x}{20}} = \frac{5}{x} \)

\( i.e. \quad \frac{1}{4} \cdot \frac{y}{x} = \frac{5}{20-x} \)

\( i.e. \quad \frac{y}{x} = \frac{5}{20-x} \)

\( y = \frac{20x}{20-x} \)

\[
\begin{array}{c|cccccc}
 x & 0 & 4 & 5 & 8 & 10 & \\
 \hline
 y & 0 & 5 & 6.25 & 13.5 & 20 & \\
\end{array}
\]

For \( x > 10, \) \( y = 20 \) we have

\( MRS_A < \frac{1}{2} \) and \( MRS_B > \frac{1}{2} \)

\( \therefore \) these allocations are PARETO efficient

3. \( x_A: \frac{1}{x_A} = \sigma_x \)

\( \frac{1}{y} = \sigma_y \)

\( x_B: \lambda_B \frac{5}{20} = \sigma_x \)

\( \therefore \lambda_B = \frac{1}{4} \)

\( y_B: \lambda_B \cdot 1 = \sigma_y \)

\( \therefore \lambda_B = \frac{4}{5} \)

\( \sigma_x = \frac{1}{4} \)

\( \sigma_y = \frac{4}{5} \)

Above establish that the FOMC are satisfied.

The complementary slackness conditions:

All three Lagrange multipliers are positive, i.e. all three constraints must be binding, which they are:

\( x_A + x_B = 20 \)

\( y_A + y_B = 20 \)

\( u_B(16, 15) = u_B(16, 15) \)
(c) We've shown in (b) that this initial allocation is Pareto efficient; therefore it's also a Walrasian equilibrium at prices proportional to the Lagrange vector at this allocation, \((\sigma_x, \sigma_y) = (\frac{4}{3}, \frac{4}{5})\). Alternatively, the equilibrium prices are in the ratio of the common MRS: \(\frac{P_x}{P_y} = \frac{5}{16}\). Let's say the price list is \((P_x, P_y) = (5, 16)\).

Now we're asked to verify by direct appeal to the definition that the equilibrium we've identified is indeed an equilibrium:

Amy is maximizing her utility:
\[
\text{MRS}_A = \frac{\frac{4}{5}}{\frac{5}{16}} = \frac{P_x}{P_y} \quad \text{and} \quad P_x x_A + P_y y_A = (5)(4) + (16)(5) = 100.
\]

Bev is maximizing her utility:
\[
\text{MRS}_B = \frac{5}{16} = \frac{P_x}{P_y} \quad \text{and} \quad P_x x_B + P_y y_B = (5)(16) + (16)(5) = 160.
\]

And markets clear:
\[
\begin{align*}
   x_A + x_B &= 4 + 16 = y_A + y_B \\
   y_A + y_B &= 5 + 15 = y_A + y_B.
\end{align*}
\]

(d) \[
\begin{align*}
   x_A: \quad \frac{1}{x_A} &= \sigma_x \quad \text{i.e.,} \quad \frac{1}{2} = \sigma_x \\
   y_A: \quad \frac{4}{y_A} &= \sigma_y \quad \text{i.e.,} \quad \frac{4}{20} = \sigma_y \\
   x_B: \quad \lambda \frac{5}{x_B} &= \sigma_x \quad \text{i.e.,} \quad \lambda \frac{5}{8} = \sigma_x \\
   y_B: \quad \lambda \cdot \frac{1}{y_B} &= \sigma_y \quad \text{i.e.,} \quad \lambda \frac{1}{15} = \frac{1}{5}, \quad \text{OK.}
\end{align*}
\]

Furthermore, all three constraints are exactly satisfied, as they must be when all Lagrange values are positive:
\[
\begin{align*}
   x_A + x_B &= 12 + 8 = 20 = x \\
   y_A + y_B &= 20 + 0 = 20 = y \\
   u_B(x_B, y_B) &= 5 \cdot \log 8 = u_B(8, 0).
\end{align*}
\]