

**Economics 501B**  
**Exercise Book**

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## The Walrasian Model and Walrasian Equilibrium

1.1 There are only two goods in the economy and there is no way to produce either good. There are  $n$  individuals, indexed by  $i = 1, \dots, n$ . Individual  $i$  owns  $\hat{x}_1^i$  units of good #1 and  $\hat{x}_2^i$  units of good #2, and his preference is described by the utility function  $u^i(x_1^i, x_2^i) = \alpha_1^i \log x_1^i + \alpha_2^i \log x_2^i$ , where  $x_1^i$  and  $x_2^i$  denote the amounts he consumes of each of the two goods, and where  $\alpha_1^i$  and  $\alpha_2^i$  are both positive. Let  $\rho$  denote the price ratio  $p_1/p_2$ . Express the equilibrium price ratio in terms of the parameters  $((\mathbf{x}^i, \alpha^i))_{i=1}^n$  that describe the economy.

1.2 Ann and Bob each own 10 bottles of beer. Ann owns 20 bags of peanuts and Bob owns no peanuts. There are no other people and no other goods in the economy, and no production of either good is possible. Using  $x$  to denote bottles of beer and  $y$  to denote bags of peanuts, Ann's and Bob's preferences are described by the following utility functions:

$$u_A(x_A, y_A) = x_A y_A^4 \quad \text{and} \quad u_B(x_B, y_B) = 2x_B + y_B.$$

Note that their *MRS* schedules are  $MRS_A = y_A/4x_A$  and  $MRS_B = 2$ .

Determine all Walrasian equilibrium price lists and allocations.

1.3 Quantities of the economy's only two goods are denoted by  $x$  and  $y$ ; no production is possible. Ann's and Ben's preferences are described by the utility functions

$$u^A(x, y) = x + y \quad \text{and} \quad u^B(x, y) = xy.$$

Ann owns the bundle (0,5) and Ben owns the bundle (30,5). Determine the Walrasian equilibrium price(s) and allocation(s).

1.4 There are two goods (quantities  $x$  and  $y$ ) and two people (Al and Bill) in the economy. Al owns eight units of the  $x$ -good and none of the  $y$ -good. Bill owns none of the  $x$ -good, and three units of the  $y$ -good. Their preferences are described by the utility functions

$$u^A(x_A, y_A) = x_A y_A \quad \text{and} \quad u^B(x_B, y_B) = y_B + \log x_B.$$

Determine the competitive equilibrium price(s) and allocation(s).

1.5 There are two consumers, Al and Bill, and two goods, the quantities of which are denoted by  $x$  and  $y$ . Al and Bill each own 100 units of the Y-good; Al owns 12 units of the X-good and Bill owns 3 units. Their preferences are described by the utility functions

$$u_A(x_A, y_A) = y_A + 60x_A - 2x_A^2 \quad \text{and} \quad u_B(x_B, y_B) = y_B + 30x_B - x_B^2.$$

Note that their marginal rates of substitution are  $MRS_A = 60 - 4x_A$  and  $MRS_B = 30 - 2x_B$ .

(a) Al proposes that he will trade *one unit* of the X-good to Bill in exchange for some units of the Y-good. Al and Bill turn to you, their economic consultant, to tell them how many units of the Y-good Bill should give to Al in order that this trade make them both strictly better off than they would be if they don't trade. What is your answer? Using marginal rates of substitution, explain how you know your answer will make them both better off.

(b) Draw the Edgeworth box diagram, including each person's indifference curve through the initial endowment point. Use different scales on the  $x$ - and  $y$ -axes or your diagram will be very tall and skinny.

(c) Determine all Walrasian equilibrium prices and allocations.

1.6 The Arrow and Debreu families live next door to one another. Each family has an orange grove that yields 30 oranges per week, and the Arrows also have an apple orchard that yields 30 apples per week. The two households' preferences for oranges ( $x$  per week) and apples ( $y$  per week) are given by the utility functions

$$u_A(x_A, y_A) = x_A y_A^3 \quad \text{and} \quad u_D(x_D, y_D) = 2x_D + y_D.$$

The Arrows and Debreus realize they may be able to make both households better off by trading apples for oranges.

Determine all Walrasian equilibrium price lists and allocations.

1.7 Amy and Bob consume only two goods, quantities of which we'll denote by  $x$  and  $y$ . Amy and Bob have the same preferences, described by the utility function

$$u(x, y) = \begin{cases} x + y - 1, & \text{if } x \geq 1 \\ 3x + y - 3, & \text{if } x \leq 1. \end{cases}$$

There are 4 units of the  $x$ -good, all owned by Amy, and 6 units of the  $y$ -good, all owned by Bob.

Draw the Edgeworth box diagram, including each person's indifference curve through the initial endowment point. Determine all Walrasian equilibrium prices and allocations.

1.8 There are  $r$  girls and  $r$  boys, where  $r$  is a positive integer. The only two goods are bread and honey, quantities of which will be denoted by  $x$  and  $y$ :  $x$  denotes loaves of bread and  $y$  denotes pints of honey. Neither the girls nor the boys are well endowed: each girl has 8 pints of honey but no bread, and each boy has 8 loaves of bread but no honey. Each girl's preference is described by the utility function  $u_G(x, y) = \min(ax, y)$  and each boy's by the utility function  $u_B(x, y) = x + y$ .

Determine the Walrasian excess demand function for honey and the Walrasian equilibrium prices and allocations.

1.9 There are only two consumers, Amy and Bev, and only two goods, the quantities of which are denoted by  $x$  and  $y$ . Amy owns the bundle  $(4, 5)$  and Bev owns the bundle  $(16, 15)$ . Amy's and Bev's preferences are described by the utility functions

$$u_A(x_A, y_A) = \log x_A + 4 \log y_A \quad \text{and} \quad u_B(x_B, y_B) = y_B + 5 \log x_B.$$

Note that the derivatives of their utility functions are

$$u_{Ax} = \frac{1}{x_A}, \quad u_{Ay} = \frac{4}{y_A}, \quad u_{Bx} = \frac{5}{x_B}, \quad u_{By} = 1.$$

Determine a Walrasian equilibrium, and verify by direct appeal to the definition that the equilibrium you have identified is indeed an equilibrium.

1.10 A consumer's preference is described by the utility function  $u(x, y) = y + \alpha \log x$  and her endowment is denoted by  $(\hat{x}, \hat{y})$ . Determine her offer curve, both analytically and geometrically.

1.11 There are two goods (quantities denoted by  $x$  and  $y$ ) and two consumers (Ann and Bob). Ann and Bob each own three units of each good. Ann's preferences are described by the relation  $MRS = y/x$  (you should be able to give a utility function that describes these preferences), but Bob's preferences are a little more complicated to describe:

If  $y < \frac{1}{2}x$ , then his indifference curve through  $(x, y)$  is horizontal.

If  $y > 2x$ , then his indifference curve through  $(x, y)$  is vertical.

If  $\frac{1}{2}x < y < 2x$ , then his  $MRS$  is  $x/y$ . (Note that this could be described by the utility function  $u(x, y) = x^2 + y^2$  in this region.)

(a) Determine Bob's offer curve, both geometrically (first) and then analytically. (Note that Bob's demand is not single-valued at a price ratio of  $\rho = 1$ .)

(b) Show that there is no price that will clear the markets – i.e., there is no Walrasian equilibrium. Do this three ways:

by drawing the aggregate offer curve,

by drawing both individual offer curves in an Edgeworth box,

and by writing the aggregate demand function analytically, and showing that at each price the market fails to clear.

1.12 The demand and supply functions for a good are

$$D(p) = \alpha + \log \frac{a}{p^2} \quad \text{and} \quad S(p) = \beta - e^{-bp}.$$

where each of the parameters  $a, b, \alpha$ , and  $\beta$  are positive. Determine how changes in the parameters will affect the equilibrium price and quantity. What is a natural interpretation of the parameter  $\beta$ ?

1.13 The demand for a particular good is given by the function  $D(p) = \alpha - 20p + 4p^2 - \frac{1}{6}p^3$  and the supply by  $S(p) = 4p$ . An equilibrium price of  $p = 6$  is observed, but then  $\alpha$  increases.

(a) Estimate the change in the equilibrium price if  $\alpha$  increases by 2.

(b) Estimate the change in the equilibrium price if  $\alpha$  increases by 1 percent.

(c) Your answers to (a) and (b) should seem a little odd. What is it that is unusual here? Draw the excess demand function, and show why this unusual result is occurring here. How many equilibria are there in this market? Which of the equilibria have this unusual feature?

1.14 The excess demand for a particular good is given by the function  $E(p) = 3 - (5 + \alpha)p + 5p^2 - p^3$  for  $p > 0$ . For all positive values of  $\alpha$ , determine how many equilibria there are and determine  $\frac{\partial p^*}{\partial \alpha}$ , where  $p^*$  denotes the equilibrium price.

1.15 Arnie has five pints of milk and no cookies. A cookie and a pint of milk are perfect substitutes to Arnie so long as he has no more than six cookies. He has no desire for more than six cookies: if he had more, he would sell or discard all but six. He always likes more milk. Bert has ten cookies and five pints of milk. He has no use for cookies — if he has cookies, he either sells them or discards them. He too always likes more milk.

(a) Provide a utility function that describes Arnie's preferences, and draw his indifference curve through the bundle  $(6, 6)$ .

(b) Determine all the Walrasian equilibrium price-lists and allocations. Verify that you've identified all the equilibria.

## Existence, Computation, and Applications of Equilibrium

2.1 Art and Bart each sell ice cream cones from carts on the boardwalk in Atlantic City. Each day they independently decide where to position their carts on the boardwalk, which runs from west to east and is exactly one mile long. Let's use  $x_A$  and  $x_B$  to denote how far (in miles) each cart was positioned yesterday from the west end of the boardwalk; and we'll use  $x'_A$  and  $x'_B$  to denote how far from the west end the carts are positioned today. Art always positions his cart as far from the boardwalk's west end as Bart's cart was *from the east end* yesterday – i.e.,  $x'_A = 1 - x_B$ . Bart always looks at  $x_A$  (how far Art was from the west end yesterday) and then positions his own cart  $x'_B = x_A^2$  miles from the west end today. Apply Brouwer's Fixed Point Theorem to prove that there is a stationary pair of locations,  $(x_A^*, x_B^*)$  – i.e., a location for each cart today that will yield the same locations again tomorrow. (This problem can be easily solved by other means – in fact, it's easy to calculate the stationary configuration. But the exercise here is to use Brouwer's Theorem.)

2.2 Two Manhattan pretzel vendors must decide where to locate their pretzel carts along a given block of Fifth Avenue. Represent the “block of Fifth Avenue” by the unit interval  $I = [0, 1] \subseteq \mathbb{R}$  – i.e., each vendor chooses a location  $x_i \in [0, 1]$ . The profit  $\pi_i$  of vendor  $i$  depends continuously on *both* vendors' locations – i.e., the function  $\pi_i : I \times I \rightarrow \mathbb{R}$  is continuous for  $i = 1, 2$ . Furthermore, each  $\pi_i$  is strictly concave in  $x_i$ .

Define an equilibrium in this situation to be a joint action  $\hat{x} = (\hat{x}_1, \hat{x}_2) \in I^2$  that satisfies both

$$\forall x_1 \in I : \pi_1(\hat{x}) \geq \pi_1(x_1, \hat{x}_2) \quad \text{and} \quad \forall x_2 \in I : \pi_2(\hat{x}) \geq \pi_2(\hat{x}_1, x_2).$$

In other words, an equilibrium consists of a location for each vendor, with the property that each one's location is best for him given the other's location.

(a) Prove that an equilibrium exists.

(b) Generalize this result to situations in which each  $\pi_i$  is merely quasiconcave – i.e., the set  $U_i(x) := \{x' \in I^2 \mid \pi_i(x') \geq \pi_i(x)\}$  is convex for each  $i$  and for every  $x \in I^2$ .

2.3 In doing applied microeconomics you often have to compute equilibria of models that don't have closed-form solutions. The computation therefore must be done by iterative numerical methods. That's what you'll do in this exercise, for the Cobb-Douglas example we analyzed in the first lecture. The iterative computation is pretty straightforward, because there are only two goods and the demand functions have simple closed-form solutions. Moreover, the equilibrium itself has a closed-form solution, so you can also have your program compute the equilibrium prices directly and then check whether your iterative program converges to the correct equilibrium prices.

Specifically, you are to use a spreadsheet program such as Excel, or a programming language such as C+ or Pascal, to compute the path taken by prices and excess demands in the two-person, two-good, pure exchange Cobb-Douglas example from the first lecture, assuming that prices adjust according to the transition function in the Arrow & Hahn proof of existence of equilibrium:

$$f(p) = \frac{1}{\sum_{k=1}^l [p_k + M_k(p)]} [p + M(p)]$$

where  $M_k(p) = \max(0, \lambda z_k(p))$  for each good  $k$ .

The proof did not actually require a  $\lambda$  — *i.e.*, we could assume that  $\lambda = 1$  — but with  $\lambda = 1$  the iterative process defined by this transition function does not converge for the Cobb-Douglas example, as you can verify once you've created your computational program. You'll find that to achieve convergence you'll need to use a  $\lambda$  equal to about .02 or smaller. Recall, too, that the proof does not actually apply to the Cobb-Douglas example, because demands are not defined for the price-lists (1,0) and (0,1). For the same reason, you can't start the iterative process off using either of these as the initial price-lists, because the "next  $p$ " defined by  $f(p)$  won't be well-defined.

You will of course have to use specific parameter values for the two consumers' utility functions and endowment bundles. With a small enough value for  $\lambda$ , the process will converge for just about any parameter values and any strictly positive initial prices. Of course, when you run your program you should note whether it does converge to the equilibrium price-list.

Plot by hand the price-line and the chosen bundles in the Edgeworth Box for several iterations of the process, or better yet, use our Edgeworth Box applet. Note that if the prices aren't sufficiently close to the equilibrium prices, the chosen bundles may not lie within the confines of the box. This is an important point to understand: each individual consumer simply takes the prices as given and chooses his or her best bundle within the resulting budget set. The consumer takes no account of the total resources available, nor of the other consumers' preferences or choices, because the consumer isn't assumed to have that information.

2.4 There are two goods and  $n$  consumers, indexed  $i = 1, \dots, n$ . Each consumer has an increasing linear preference – *i.e.*, each consumer’s preference is described by a utility function of the form

$$u^i(x_1, x_2) = a^i x_1 + b^i x_2,$$

where  $a^i$  and  $b^i$  are positive numbers. No production of either good is possible, but each consumer owns positive amounts of each good.

(a) Prove that this economy has a Walrasian equilibrium.

(b) Is the equilibrium price ratio unique? Is the equilibrium allocation unique?

**Helpful Hint:** Look for ways to make this problem tractable. For example, it might be helpful to index the consumers according to the slopes of their indifference curves – *e.g.*, the flattest as  $i = 1$ , the next flattest as  $i = 2$ , and so on. Also, do you need both preference parameters? And it might be easier to work it out first for  $n = 2$  and perhaps with each consumer owning the same amount of each good.

You’ll probably find it helpful to use the following two theorems on the sum and composition of correspondences that have closed graphs:

**Theorem:** If  $Y$  is compact and the correspondences  $f : X \rightrightarrows Y$  and  $g : X \rightrightarrows Y$  both have closed graphs, then the sum  $f + g$  also has a closed graph, where  $f + g$  is the correspondence defined by

$$(f + g)(x) := \{ y_1 + y_2 \in Y \mid y_1 \in f(x) \text{ and } y_2 \in g(x) \}.$$

**Theorem:** If  $Y$  and  $Z$  are compact and the correspondences  $f : X \rightrightarrows Y$  and  $g : Y \rightrightarrows Z$  both have closed graphs, then the composition  $f \circ g$  also has a closed graph, where  $f \circ g$  is the correspondence defined by

$$(f \circ g)(x) := g(f(x)) := \{ z \in Z \mid \exists y \in Y : y \in f(x) \text{ \& } z \in g(y) \} = \bigcup \{ g(y) \mid y \in f(x) \}.$$

2.5 In our proof of the existence of a Walrasian equilibrium, the following sentences appear: “We know that  $\widehat{\zeta}$  has a closed graph and is non-empty-valued and convex-valued, and it is easy to show that  $\mu$  has the same properties. Therefore so does  $f$ , and Kakutani’s Theorem therefore implies that  $f$  has a fixed point.” For this exercise, use the definitions of  $\widehat{\zeta}$ ,  $\mu$ , and  $f$  given in the existence proof. The following proofs are all elementary: the key is understanding the concepts of correspondence, a closed set, a convex set, and the product of two sets. The point of this exercise is to work with those concepts.

(a) If  $l = 2$ , then  $\mu$  can be written as follows:

$$\mu(z_1, z_2) = \begin{cases} S, & \text{if } z_1 = z_2 \\ \{(1, 0)\}, & \text{if } z_1 > z_2 \\ \{(0, 1)\}, & \text{if } z_1 < z_2 \end{cases}$$

For this  $l = 2$  case, prove that  $\mu$  has a closed graph.

(b) Write a detailed definition of  $\mu$  for the case  $l = 3$ , like the one above for  $l = 2$ .

(c) It’s obvious in (a) and (b) – assuming you’ve written (b) correctly – that  $\mu$  is convex-valued. Give a single proof, for all values of  $l$ , that  $\mu$  is convex-valued.

(d) Prove that if  $\widehat{\zeta}$  and  $\mu$  both have closed graphs, then so does  $f$ .

(e) Prove that if  $\widehat{\zeta}$  and  $\mu$  are both convex-valued, then so is  $f$ .

2.6 As in Harberger's example, assume that there are two goods produced: product  $X$  is produced by firms in the "corporate" sector and product  $Y$  by firms in the "non-corporate" sector. Both products are produced using the two inputs **labor** and **capital** (quantities denoted by  $L$  and  $K$ ). Production functions are

$$X = \sqrt{L_X K_X} \quad \text{and} \quad Y = \sqrt{L_Y K_Y}.$$

All consumers have preferences described by the utility function  $u(X, Y) = XY$ . The consumers care only about consuming  $X$  and  $Y$ , and they supply labor and capital inelastically in the total amounts  $L = 600$  and  $K = 600$ . Let  $p_X, p_Y, p_L$ , and  $p_K$  denote the prices of the four goods in the economy; assume that  $p_L = 1$  always.

(a) What is the Walrasian equilibrium?

(b) Suppose that a 50% tax is imposed on payments to capital in the corporate sector only, and that the government uses the tax proceeds to purchase equal amounts of the output of the two sectors. What will be the new Walrasian equilibrium? How is welfare affected by the tax – are people better off with or without the tax?

2.7 Many applications of microeconomic theory use the concept of a representative consumer. In order for this concept to be meaningful, as we've seen, the economy must satisfy rather special conditions. For this exercise assume there are two consumers,  $i = 1, 2$ , whose utility functions are  $u_i(x_i, y_i) = x_i^{\alpha_i} y_i^{\beta_i}$  and whose initial holdings are  $(\hat{x}_1, \hat{y}_1)$  and  $(\hat{x}_2, \hat{y}_2)$ . Assume that

$$\frac{\hat{x}_1}{\hat{x}_1 + \hat{x}_2} = \frac{\hat{y}_1}{\hat{y}_1 + \hat{y}_2} = \lambda_1 \quad \text{and} \quad \frac{\hat{x}_2}{\hat{x}_1 + \hat{x}_2} = \frac{\hat{y}_2}{\hat{y}_1 + \hat{y}_2} = \lambda_2.$$

According to Eisenberg's Theorem, the market demand function is also the demand function of the ("representative") consumer with initial holdings  $(\hat{x}_1 + \hat{x}_2, \hat{y}_1 + \hat{y}_2)$  and with utility function

$$u(x, y) = \max\{ u_1(x_1, y_1)^{\lambda_1} u_2(x_2, y_2)^{\lambda_2} \mid x_1 + x_2 = x, y_1 + y_2 = y \}.$$

Show that  $u(x, y) = x^\alpha y^\beta$ , where  $\alpha = \lambda_1 \alpha_1 + \lambda_2 \alpha_2$  and  $\beta = \lambda_1 \beta_1 + \lambda_2 \beta_2$ .

2.8 Consider the following five sets:

$$A = \{ x \in \mathbb{R}^2 \mid x_1, x_2 \geq 0 \text{ and } x_1^2 + x_2^2 = 1 \}$$

$$B = \{ x \in \mathbb{R}^2 \mid x_1, x_2 \geq 0 \text{ and } 1 \leq x_1^2 + x_2^2 \leq 2 \}$$

$$C = \{ x \in \mathbb{R}^2 \mid 1 \leq x_1^2 + x_2^2 \leq 2 \}$$

$$D = \{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \}$$

$$E = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1 \}$$

(a) Draw a diagram of each set.

(b) To which sets does Brouwer's Fixed Point Theorem apply? (That is, which sets satisfy the assumptions of the theorem?)

(c) Which sets admit a counterexample to Brouwer's Theorem? (That is, for which sets is it possible to define a continuous function  $f$  mapping the set into itself for which  $f$  has no fixed point?)

(d) For each of the sets you've identified in (c), provide a continuous function  $f$  that has no fixed point.

2.9 Two stores, Una Familia and Dos Hijos, are in the same neighborhood and compete in selling a particular product. Every Tuesday, Thursday, and Saturday Dos Hijos changes its posted price  $p_2$  in response to the price  $p_1$  Una Familia charged on the previous day, according to the continuous function  $p_2 = f_2(p_1)$ . Every Wednesday and Friday Una Familia changes its posted price in response to the price  $p_2$  Dos Hijos charged on the previous day, according to the continuous function  $p_1 = f_1(p_2)$ . The stores are closed on Sunday; on Monday Una Familia responds to Dos Hijos's preceding Saturday price, also according to  $f_1(\cdot)$ . Una Familia cannot sell any units at a price above  $\bar{p}_1$ , no matter what price Dos Hijos charges, so Una Familia never charges a price higher than  $\bar{p}_1$ . Similarly, Dos Hijos never charges a price higher than  $\bar{p}_2$ . Prove that there is an "equilibrium" pair of prices  $(p_1^*, p_2^*)$  — prices that can persist day after day, week after week, with neither store changing its price.

2.10 (Bewley) A securities analyst publishes a forecast of the prices of  $n$  securities. She knows that the prices  $p_k$  of the securities are influenced by her forecast according to the continuous function  $(p_1, \dots, p_n) = f(q_1, \dots, q_n)$ , where  $q_k$  is her forecast of the price  $p_k$ . Whatever prices she forecasts, none of the realized prices ever exceeds  $Q$  — *i.e.*, there is a (large) number  $Q$  such that  $\forall \mathbf{q} \in \mathbb{R}^n : f_k(\mathbf{q}) \leq Q$  for  $k = 1, \dots, n$ .

- (a) Prove that there exists a forecast  $\mathbf{q}^* = (q_1^*, \dots, q_n^*)$  that will turn out to be perfectly accurate.
- (b) The analyst can write down the functions  $f_k(q_1, \dots, q_n)$  for every  $k$ , but she can't solve the system of equations  $f(\mathbf{q}) = \mathbf{q}$  analytically. Describe a method by which she might be able to arrive at an accurate forecast.

2.11 This exercise builds on Exercise 2.3. Replace the two Cobb-Douglas consumers of #2.3 with consumers whose utility functions have the form  $u(x_1, x_2) = 2\sqrt{\alpha x_1} + 2\sqrt{\beta x_2}$ .

(a) Derive the consumers' demand functions. You should obtain

$$x_1 = \frac{\alpha}{\beta p_1 + \alpha p_2} \left( \frac{p_2}{p_1} \right) W \quad \text{and} \quad x_2 = \frac{\beta}{\beta p_1 + \alpha p_2} \left( \frac{p_1}{p_2} \right) W,$$

where  $W = p_1 \dot{x}_1 + p_2 \dot{x}_2$  is the consumer's wealth. Therefore

$$x_1 - \dot{x}_1 = \frac{1}{\beta p_1 + \alpha p_2} \left( \frac{1}{p_1} \right) (\alpha p_2^2 \dot{x}_2 - \beta p_1^2 \dot{x}_1) \quad \text{and} \quad x_2 - \dot{x}_2 = \frac{1}{\beta p_1 + \alpha p_2} \left( \frac{1}{p_2} \right) (\beta p_1^2 \dot{x}_1 - \alpha p_2^2 \dot{x}_2).$$

(b) Assume that each consumer's  $\alpha$  is the same and each consumer's  $\beta$  is the same. Verify that the market excess demand functions for the two goods are the demands of a fictitious "representative consumer" whose utility function is the one given above and whose endowment bundle  $(\dot{x}_1, \dot{x}_2)$  is the sum of the two actual consumers' endowments:  $(\dot{x}_1, \dot{x}_2) = (\dot{x}_1^1, \dot{x}_2^1) + (\dot{x}_1^2, \dot{x}_2^2)$ , where  $\dot{x}_k^i$  denotes consumer  $i$ 's endowment of good  $k$ . Determine the equilibrium price-ratio.

(c) Now assume that Consumer 1's utility parameters have the values  $\alpha^1 = 2$  and  $\beta^1 = 1$  and that Consumer 2's are  $\alpha^2 = 1$  and  $\beta^2 = 1$ . Assume that  $\dot{x}_k^i = 40$  for each  $i$  and  $k$ . In this case the market demand functions are no longer those of a representative consumer, and the equilibrium condition (*viz.* that market excess demand is zero) is a third-degree polynomial equation which would be difficult to solve analytically. (It could be solved numerically, but if there were more consumers, the equilibrium equation would be even more complicated to solve. And if there were more goods — and thus more price variables and more equations to characterize equilibrium — it would require a very complex numerical procedure to calculate the equilibrium prices directly from the equilibrium equations.) But the computational program you developed in Exercise 2.3 can easily be adapted to calculate the equilibrium price-ratio. How small do you find you must make the price-adjustment parameter  $\lambda$  in order to get the prices to converge? When you get them to converge you should find that the equilibrium price-ratio is approximately  $p_1/p_2 = 1.186141$ .

## Pareto Improvements and Pareto Efficiency

3.1 Assume throughout this exercise that

$P$  is an irreflexive relation on a set  $X$

$P^c$  denotes the complement of  $P$  — *i.e.*,  $xP^cy$  if and only if “not  $xPy$ ”

$I := P^c \cup (P^{-1})^c$  — *i.e.*,  $xIy$  if and only if neither  $xPy$  nor  $yPx$

$R := P \cup I$  — *i.e.*,  $xRy$  if and only if  $xPy$  or  $xIy$ .

For any list  $(P_1, \dots, P_n)$  of preference relations, let  $\bar{P}$  denote the associated Pareto relation, *i.e.*, the Pareto aggregation of  $(P_1, \dots, P_n)$ .

- (a) Prove that the relation  $R$  is transitive if and only if its associated  $P$  and  $I$  are both transitive.
- (b) Prove that if, for each  $i \in N = \{1, \dots, n\}$ ,  $P_i$  is irreflexive, then  $\bar{P}$  is irreflexive.
- (c) Provide a counterexample to the following proposition: “If, for each  $i \in N$ ,  $P_i$  is transitive, then  $\bar{P}$  is transitive.” (Try to find the simplest possible counterexample. It might help to use the interpretation that the elements of  $X$  are universities, or economics departments, or basketball teams, etc. It may also help in this case to remember that a binary relation on a set  $X$  is a subset of  $X \times X$ .)
- (d) Prove that if, for an irreflexive relation  $P$ , the associated  $R$  is transitive, then
  - (i)  $xPy \ \& \ yRz \Rightarrow xPz$
  - (ii)  $xRy \ \& \ yPz \Rightarrow xPz$ .
- (e) Prove that if, for each  $i \in N$ ,  $R_i$  is transitive, then  $\bar{P}$  is transitive.

The lecture notes provide examples which show that if each  $R_i$  is transitive,  $\bar{I}$  need not be transitive, and thus, according to (a) above,  $\bar{R}$  need not be transitive.

3.2 This exercise requires only the definition of Pareto efficiency and the marginal condition that characterizes interior Pareto allocations when there are only two goods – *viz.* that each person has the same marginal rate of substitution between the two goods. (Recall that an interior allocation is one in which everyone’s bundle includes a positive amount of each good.)

(a) Show diagrammatically that the following statement is true for any bundle  $(x, y) \in \mathbb{R}^2$  at which a consumer’s marginal rate of substitution is defined:

(\*) A change  $(\Delta x, \Delta y)$  in the bundle will make the consumer worse off if  $\Delta y < (MRS)(-\Delta x)$ .

(b) In Exercise 1.6 determine all interior Pareto allocations and depict them in an Edgeworth box diagram.

(c) Someone has proposed that the endowment  $(\dot{x}, \dot{y}) = (60, 30)$  of apples and oranges from the two families’ orchards be allocated as follows: the Arrow family would receive the bundle  $(\hat{x}_A, \hat{y}_A) = (20, 30)$  and the Debreu family would receive the bundle  $(\hat{x}_D, \hat{y}_D) = (40, 0)$ . In the Edgeworth box, depict each household’s indifference curve through the proposed allocation. Use the definition of Pareto efficiency and the condition (\*) above to verify that this proposal is Pareto efficient in spite of the fact that  $MRS_A < MRS_D$  at the proposed allocation.

(d) In Exercise 1.4 determine all interior Pareto allocations and depict them in an Edgeworth box diagram. Does the argument in (c) work here for the allocation in which  $(x_A, y_A) = (2, 3)$  and  $(x_B, y_B) = (6, 0)$ ?

3.3 There are only two goods and two consumers in the economy, and no production is possible. The consumers’ preferences can be represented by the utility functions

$$u^1(x, y) = y + \log(1 + x) \quad \text{and} \quad u^2(x, y) = y + 2 \log(1 + x).$$

for all bundles in which  $x, y \geq 0$ . Each consumer is endowed with 5 units of each good. Determine all interior Pareto allocations and depict them in an Edgeworth box diagram. Consider all the allocations in which  $y_A = 10$  and  $y_B = 0$ ; to which of these allocations does the “boundary” argument in Exercise 3.2 apply? Can you make a similar argument about any of the allocations in which  $y_A = 0$  and  $y_B = 10$ ?

3.4 (See Exercise 1.5) There are two consumers, Al and Bill, and two goods, the quantities of which are denoted by  $x$  and  $y$ . Al and Bill each own 100 units of the Y-good; Al owns 12 units of the X-good and Bill owns 3 units. Their preferences are described by the utility functions

$$u_A(x_A, y_A) = y_A + 60x_A - 2x_A^2 \quad \text{and} \quad u_B(x_B, y_B) = y_B + 30x_B - x_B^2.$$

Note that their marginal rates of substitution are  $MRS_A = 60 - 4x_A$  and  $MRS_B = 30 - 2x_B$ .

Determine the entire set of Pareto allocations. (You may do this via MRS conditions.) Depict the set in an Edgeworth box diagram. (Use different scales on the  $x$ - and  $y$ -axes or your diagram will be very tall and skinny.)

3.5 Ann and Bill work together as water ski instructors in Florida. Each earns \$100 per day. Each one also owns orange trees that yield 8 oranges per day. Ann likes oranges “more” than Bill does; specifically, Ann’s MRS for oranges is  $MRS_A = 12 - x_A$  and Bill’s MRS is  $MRS_B = 8 - x_B$ , where  $x_i$  denotes  $i$ ’s daily consumption of oranges and the MRS tells how many dollars (i.e., how much consumption of other goods) one would be willing to give up to get an additional orange.

(a) Bill has been selling two oranges a day to Ann, for which Ann has been paying Bill \$3 per day. (Thus, Ann ends up with 10 oranges and \$97 per day, and Bill ends up with 6 oranges and \$103 per day.) Is this Pareto efficient? Are they both better off than they would be if they did not trade? Is this a Walrasian Equilibrium? Verify your answers.

(b) In an Edgeworth box diagram depict clearly *all* Pareto efficient allocations of oranges and dollars to Ann and Bill.

(c) A hurricane has destroyed Ann’s orange crop but has left Bill’s crop undamaged. The Florida legislature has hurriedly passed a law against “price gouging.” The law specifies that oranges cannot be sold for more than four dollars apiece. At the price of four dollars, Bill is willing to sell Ann four oranges per day, but not more. Would Ann be willing to buy four oranges at four dollars apiece? Are there illegal trades (i.e., at a price of more than four dollars per orange) that would make them both better off than they are at the legal trade of four oranges for four dollars apiece? If so, find such a trade; if not, explain why not.

3.6 (See Exercise 1.2) Ann and Bob each own 10 bottles of beer. Ann owns 20 bags of peanuts and Bob owns no peanuts. There are no other people and no other goods in the economy, and no production of either good is possible. Using  $x$  to denote bottles of beer and  $y$  to denote bags of peanuts, Ann's and Bob's preferences are described by the following utility functions:

$$u_A(x_A, y_A) = x_A y_A^4 \quad \text{and} \quad u_B(x_B, y_B) = 2x_B + y_B.$$

Note that their *MRS* schedules are  $MRS_A = y_A/4x_A$  and  $MRS_B = 2$ .

- (a) Determine all Walrasian equilibrium price lists and allocations.
- (b) Determine all core allocations. (This is Exercise 5.16.)
- (c) Determine all boundary allocations that are Pareto efficient.
- (d) Determine all interior allocations that are Pareto efficient, and draw the set of all Pareto efficient allocations in an Edgeworth box.
- (e) Give two alternative utility functions that describe Ann's preferences, one that is strictly concave and additively separable (i.e., of the form  $u(x, y) = v(x) + w(y)$ ), and one that is neither strictly concave nor additively separable.

3.7 There are two goods (quantities  $x$  and  $y$ ) and two people (Ann and Bob) in the economy. Ann owns two units of each good and Bob owns six units of each good. Their preferences are described by the utility functions:

$$u^A(x_A, y_A) = x_A^2 y_A \quad \text{and} \quad u^B(x_B, y_B) = y_B - \frac{1}{2}(8 - x_B)^2.$$

(a) Derive the complete marginal conditions that characterize the Pareto optimal allocations, and draw the set of all Pareto allocations in an Edgeworth box diagram.

(b) Determine the competitive equilibrium price(s) and allocation(s).

(c) For each of the following allocations determine whether the allocation is Pareto optimal. If it is, give all the “decentralizing” price lists; if it’s not, find a Pareto optimal allocation that makes Ann and Bob both strictly better off.

$$(c1) \quad (x_A, y_A) = (6, 8), \quad (x_B, y_B) = (2, 0)$$

$$(c2) \quad (x_A, y_A) = (8, 2), \quad (x_B, y_B) = (0, 6)$$

$$(c3) \quad (x_A, y_A) = (4, 8), \quad (x_B, y_B) = (4, 0)$$

3.8 There are two goods (quantities  $x$  and  $y$ ) and two people (Andy and Bea) in the economy. No production is possible. An allocation is a list  $(x_A, y_A, x_B, y_B)$  specifying what each person receives of each good. Andy’s and Bea’s preferences are described by the utility functions

$$u^A(x_A, y_A) = 2x_A + y_A + \alpha \log x_B \quad \text{and} \quad u^B(x_B, y_B) = x_B + y_B.$$

The two goods are available in the positive amounts  $\hat{x}$  and  $\hat{y}$ , and  $\alpha$  satisfies  $0 < \alpha < \hat{x}$ . Note that Andy cares directly about how much Bea receives of the  $x$ -good.

Determine all the Pareto efficient allocations in which Andy and Bea both receive a positive amount of each good.

3.9 (See Exercise 1.8) There are  $r$  girls and  $r$  boys, where  $r$  is a positive integer. The only two goods are bread and honey, quantities of which will be denoted by  $x$  and  $y$ :  $x$  denotes loaves of bread and  $y$  denotes pints of honey. Neither the girls nor the boys are well endowed: each girl has 8 pints of honey but no bread, and each boy has 8 loaves of bread but no honey. Each girl's preference is described by the utility function  $u_G(x, y) = \min(ax, y)$  and each boy's by the utility function  $u_B(x, y) = x + y$ .

(a) Determine the Walrasian excess demand function for honey and the Walrasian equilibrium prices and allocations.

(b) Determine the set of Pareto optimal allocations for  $r = 1$  and for arbitrary  $r$ .

(c) Assume that  $a = 1$ . Determine the core allocations for  $r = 1$ , for  $r = 2$ , and for arbitrary  $r$ .

3.10 Amy owns five bottles of wine, but no cheese. Bob owns ten pounds of cheese, but no wine. Their preferences for wine and cheese are described by the following marginal rates of substitution ( $x$  denotes wine consumption, in bottles, and  $y$  denotes cheese consumption, in pounds):

$$\text{Amy: } MRS_A = \begin{cases} 5, & \text{if } x < 3 \\ 1, & \text{if } x > 3 \end{cases} \quad \text{Bob: } MRS_B = 6 - x.$$

(a) Draw Amy's indifference curve that contains the bundle (3,3). Is Amy's preference representable by a continuous utility function? If so, give such a function; if not, indicate why not. Draw Bob's indifference curve through the bundle (4,2).

(b) In an Edgeworth box diagram, depict the entire set of Pareto optimal allocations

(c) Determine all Walrasian equilibrium price lists and allocations.

(d) Suppose Amy and Bob are joined by Ann and Bill. Ann is exactly like Amy (same preferences, same endowment), and Bill is exactly like Bob. So, now there are two people of each type. Show that the following allocation is not in the core: Amy and Ann each get (3,2), and Bob and Bill each get (2,8).

3.11 There are two goods (quantities  $x$  and  $y$ ) and two people (Amy and Bev) in the economy. No production is possible. There are 30 units of the  $x$ -good and 60 units of the  $y$ -good available to be distributed to Amy and Bev, whose preferences are as follows:

Amy's MRS is 3 if  $y > x$  and her MRS is  $1/2$  if  $y < x$ ;

Bev's MRS is always 1.

(a) Draw an Edgeworth box diagram and indicate on the diagram the entire set of Pareto optimal allocations.

(b) If Amy owns the bundle (20,60) and Bev owns the bundle (10,0), determine the competitive (Walrasian) equilibrium price(s) and allocation(s).

3.12 (See Exercise 1.9) There are only two consumers, Amy and Bev, and only two goods, the quantities of which are denoted by  $x$  and  $y$ . There are 20 units of each good to be allocated between Amy and Bev. Amy's and Bev's preferences can be represented by the utility functions

$$u_A(x_A, y_A) = \log x_A + 4 \log y_A \quad \text{and} \quad u_B(x_B, y_B) = y_B + 5 \log x_B.$$

(a) Determine the set of all Pareto allocations and depict the set carefully in an Edgeworth box diagram. (You may do this via MRS conditions.)

(b) Verify that the allocation  $((x_A, y_A), (x_B, y_B)) = ((4, 5), (16, 15))$  is Pareto efficient by finding values of the Lagrange multipliers in the first-order conditions for the problem (P-max) and then showing that with these Lagrange values the first-order conditions are indeed satisfied.

(c) Now assume that Amy owns the bundle (4, 5) and Bev owns the bundle (16, 15). Determine a Walrasian equilibrium, and verify by direct appeal to the definition that the equilibrium you have identified is indeed an equilibrium.

(d) Verify that the allocation  $((x_A, y_A), (x_B, y_B)) = ((12, 20), (8, 0))$  is Pareto efficient by finding values of the Lagrange multipliers in the first-order conditions for the problem (P-max) and then showing that with these Lagrange values the first-order conditions are indeed satisfied.

3.13 (See Exercises 1.4 and 3.2) There are two goods (quantities  $x$  and  $y$ ) and two people (Al and Bill) in the economy. Al owns eight units of the  $x$ -good and none of the  $y$ -good. Bill owns none of the  $x$ -good, and three units of the  $y$ -good. Their preferences are described by the utility functions

$$u^A(x_A, y_A) = x_A y_A \quad \text{and} \quad u^B(x_B, y_B) = y_B + \log x_B.$$

- (a) Determine the competitive equilibrium price(s) and allocation(s).
- (b) Derive the complete marginal conditions that characterize the Pareto optimal allocations, and draw the set of all Pareto optimal allocations in an Edgeworth box diagram.
- (c) For each of the following allocations determine whether the allocation is Pareto optimal. If it is, give all the “decentralizing” price lists; if it isn’t, find a Pareto optimal allocation that makes both Al and Bill strictly better off.

$$(c1) \quad (x_A, y_A) = (4, 1), \quad (x_B, y_B) = (4, 2)$$

$$(c2) \quad (x_A, y_A) = (1, 3), \quad (x_B, y_B) = (7, 0)$$

$$(c3) \quad (x_A, y_A) = (4, 2), \quad (x_B, y_B) = (2, 1)$$

$$(c4) \quad (x_A, y_A) = (7, 3), \quad (x_B, y_B) = (1, 0)$$

3.14 (See Exercise 1.8) Quantities of the economy’s only two goods are denoted by  $x$  and  $y$ ; no production is possible. Ann’s and Ben’s preferences are described by the utility functions

$$u^A(x, y) = ax + y \quad \text{and} \quad u^B(x, y) = x^b y.$$

- (a) Let  $w_x$  and  $w_y$  denote the available amounts of the two goods. Determine all the Pareto efficient allocations, expressing them in terms of the parameters  $a, b, w_x$ , and  $w_y$ . For each of following three cases, draw an Edgeworth box diagram and indicate on the diagram the entire set of Pareto efficient allocations:

$$\text{Case I: } \frac{w_y}{w_x} = \frac{a}{b} \quad \text{Case II: } \frac{w_y}{w_x} > \frac{a}{b} \quad \text{Case III: } \frac{w_y}{w_x} < \frac{a}{b}.$$

- (b) Let  $a = b = 1$ , and suppose that Ann owns the bundle  $(0, 5)$  and Ben owns the bundle  $(30, 5)$ . Determine the Walrasian equilibrium price(s) and allocation(s).

3.15 There are two goods (quantities  $x$  and  $y$ ) in the economy and two people, Alex and Beth, whose preferences are described by the utility functions

$$u^A(x_A, y_A) = x_A + 2y_A \quad \text{and} \quad u^B(x_B, y_B) = y_B - \frac{1}{2}(12 - x_B)^2.$$

Let  $\hat{x}_i$  and  $\hat{y}_i$  denote  $i$ 's initial holdings ( $i = A, B$ ), and assume that between them Alex and Beth own a total of 10 units of each good. Let  $r$  denote the ratio  $\hat{y}_B/\hat{x}_A$ , and consider the following three cases:

$$\text{Case I: } r > 2 \quad \text{Case II: } \frac{1}{2} < r < 2 \quad \text{Case III: } r < \frac{1}{2}.$$

- (a) Assuming we're in Case I, determine the complete first-order conditions that characterize the Pareto optimal allocations in terms of marginal rates of substitution. Draw the set of all Pareto optimal allocations in an Edgeworth box diagram.
- (b) Describe informally how the set of Pareto optimal allocations and first-order conditions in (a) are changed if we're in Case II or Case III.
- (c) Assuming that each person owns five units of each good before trading, determine all the competitive equilibrium price(s) and allocation(s).
- (d) Assuming that Beth owns all ten units of the  $y$ -good, and that each person owns five units of the  $x$ -good, determine all the competitive equilibrium price(s) and allocation(s).
- (e) Determine whether it is Pareto optimal for Alex to be given all of the  $y$ -good and Beth all of the  $x$ -good. If so, determine all the decentralizing prices; if not, find a Pareto improvement.
- (f) Determine the competitive equilibrium prices in Case I, Case II, and Case III.

3.16 (See Exercise 1.6) Amy and Bob consume only two goods, quantities of which we'll denote by  $x$  and  $y$ . Amy and Bob have the same preferences, described by the utility function

$$u(x, y) = \begin{cases} x + y - 1, & \text{if } x \geq 1 \\ 3x + y - 3, & \text{if } x \leq 1. \end{cases}$$

There are 4 units of the  $x$ -good, all owned by Amy, and 6 units of the  $y$ -good, all owned by Bob.

(a) Draw the Edgeworth box diagram, including each person's indifference curve through the initial endowment point. Determine all Walrasian equilibrium prices and allocations.

(b) In an Edgeworth box diagram, depict all Pareto optimal allocations.

(c) In an Edgeworth box diagram, depict all core allocations.

Suppose Cal joins Amy and Bob. Cal owns 18 units of the  $y$ -good but none of the  $x$ -good, and he has preferences described by the utility function  $u(x, y) = 2x + y$ .

(d) Determine all competitive equilibrium prices and allocations.

(e) Show that now, with Cal present, none of the core allocations give Amy and Bob what they received in any of the competitive allocations in (a).

3.17 The economy consists of two people (Mr. A and Mr. B) and two goods (the quantities of which will be denoted by  $x$  and  $y$ ). Mr. A owns all the  $x$ -good (4 units) and Mr. B owns all the  $y$ -good (6 units). It is not possible to produce any additional units of either good. Let  $(x_i, y_i)$  denote the bundle allocated to (or consumed by) Mr.  $i$ , where  $i$  may be either  $A$  or  $B$ . The two people's preferences are described by the following utility functions

$$\begin{aligned}
 u_A(x_A, y_A) &= \begin{cases} y_A + 3x_A, & \text{if } x \leq 2 \\ y_A + \frac{1}{2}x_A + 5, & \text{if } x \geq 2 \end{cases} \\
 u_B(x_B, y_B) &= y_B - \frac{1}{2}(4 - x_B)^2.
 \end{aligned}$$

- (a) Depict the set of all Pareto optimal allocations in an Edgeworth box diagram.
- (b) Determine all the Walrasian equilibrium price lists and allocations, and depict them in an Edgeworth box diagram.
- (c) Suppose that another person just like Mr.  $A$  (same preferences, same endowment) is added to the economy, and also another person just like Mr.  $B$ . (So now there are two people of each type.) Show that the following allocation is not in the core: each type- $A$  person gets  $(2, 1)$  and each type- $B$  person gets  $(2, 5)$ .

3.18 The following theorem appears in the lecture notes: If every  $u^i$  is continuous and locally nonsatiated, then an interior allocation  $\hat{\mathbf{x}}$  is Pareto efficient for the economy  $(u^i, \hat{\mathbf{x}}^i)_1^n$  if and only if it is a solution of the problem (P-Max).

- (a) Provide a counterexample to show why, for interior allocations, the theorem requires that utility functions be locally nonsatiated.
- (b) Provide a counterexample to show why, at a boundary allocation, local nonsatiation is not enough — a boundary allocation could be a solution of (P-Max) but not Pareto efficient, even if every  $u^i$  is continuous and locally nonsatiated.

3.19 There are two goods (quantities are denoted by  $x$  and  $y$ ) and two people (Alex and Beth), whose preferences are described by the utility functions

$$u^A(x, y) = xy \quad \text{and} \quad u^B(x, y) = 2x + y.$$

There are eight units of the  $x$ -good to be allocated and six units of the  $y$ -good. Someone has proposed that the bundles  $(x_A, y_A) = (2, 4)$  and  $(x_B, y_B) = (6, 2)$  be allocated to Alex and Beth.

(a) Determine the gradients  $\nabla u^A$  and  $\nabla u^B$  at the proposal. Draw Alex's and Beth's consumption spaces, including the bundles they would receive in the proposal, their indifference curves through those bundles, and the gradients at those bundles. Is  $\nabla u^A = \lambda \nabla u^B$  for some  $\lambda$ ?

(b) Write down the Pareto maximization problem (P-Max), obtain the first-order marginal conditions (FOMC), and then evaluate the first-order conditions at the proposal. (Use the notation  $\sigma_x$  and  $\sigma_y$  for the Lagrange multipliers associated with the feasibility constraints, and  $\lambda$  for the Lagrange multiplier associated with the constraint on Beth's utility level.) Determine whether the proposal satisfies the FOMC — *i.e.*, determine whether there are values of the three Lagrange multipliers for which the FOMC are satisfied at the proposal.

(c) Determine whether each gradient  $\nabla u^i$  is a multiple of the vector  $(\sigma_x, \sigma_y)$ .

(d) Determine Alex's and Beth's marginal rates of substitution at the proposal.

(e) Someone else has proposed that the bundles  $(x_A, y_A) = (6, 2)$  and  $(x_B, y_B) = (2, 4)$  be allocated to Alex and Beth, the reverse of the first proposal. Answer the same questions (a)-(d) for this second proposal.

(f) Determine all the interior Pareto allocations to Alex and Beth. Draw the set of these allocations in an Edgeworth box diagram.

(g) Consider a third proposal,  $(x_A, y_A) = (6, 6)$  and  $(x_B, y_B) = (2, 0)$ . Determine whether the FOMC for the problem (P-Max) are satisfied for this proposal.

3.20 One possible social welfare criterion for choosing among alternative allocations is the sum of individuals' utilities, or a weighted sum of the utilities:

$$W(\mathbf{x}^1, \dots, \mathbf{x}^n) = \sum_{i=1}^n \theta_i u^i(\mathbf{x}^i),$$

where  $\theta_1, \dots, \theta_n$  are exogenously given weights (positive real numbers).

Assume there are just two goods and two consumers, with utility functions of the form

$$u_i(x, y) = \alpha_i \log x + \beta_i \log y.$$

Assume that  $\dot{x}$  and  $\dot{y}$  are the total amounts of the goods that are available to distribute to the two consumers.

(a) Determine the allocation(s) that maximize  $W(\cdot)$  as a function of the eight parameters  $\theta_1, \theta_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \dot{x}$ , and  $\dot{y}$ .

Solution:

$$x_i = \frac{\theta_i \alpha_i}{\theta_1 \alpha_1 + \theta_2 \alpha_2} \dot{x} \quad \text{and} \quad y_i = \frac{\theta_i \beta_i}{\theta_1 \beta_1 + \theta_2 \beta_2} \dot{y}, \quad i = 1, 2.$$

(b) Determine which, if any, of the allocations that maximize  $W(\cdot)$  also satisfy the condition  $MRS_1 = MRS_2$ .

(c) Assume that  $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 1$  and  $(\dot{x}, \dot{y}) = (30, 60)$ . What is the “welfare maximizing” allocation if  $\theta_1 = \theta_2$ ? Depict this situation in an Edgeworth box diagram. As the  $\theta$ 's vary over all possible values, determine the set of allocations that could possibly maximize welfare for *some* value(s) of  $\theta$ .

(d) Assume that  $\alpha_1 = \beta_1 = 1, \alpha_2 = \beta_2 = 2$ , and  $(\dot{x}, \dot{y}) = (30, 60)$ . What is the “welfare maximizing” allocation if  $\theta_1 = \theta_2$ ? Depict this situation in an Edgeworth box diagram. As the  $\theta$ 's vary over all possible values, determine the set of allocations that could possibly maximize welfare for *some* value(s) of  $\theta$ .

(e) Compare the allocation in (c) to the allocation in (d) for arbitrary values of  $\theta_1$  and  $\theta_2$ .

(f) How do the consumers' indifference maps in (d) differ from their maps in (c)?

3.21 In Exercise #1.15 identify all the Pareto allocations.

## Adding Production to the Model

4.1 There are two people (A and B) and two goods (wheat and bread). One production process is available, a process by which one bushel of wheat can be turned into two loaves of bread. The individuals' preferences are described by the utility functions

$$u^A(x, y) = xy \quad \text{and} \quad u^B(x, y) = x^2y.$$

where  $x$  is the person's consumption of wheat (in bushels) and  $y$  is the person's consumption of bread (in loaves). The two people are endowed with a total of 60 bushels of wheat and no bread. For the consumption allocations in (a), (b), and (c) below, do the following: if the given allocation is Pareto optimal, then verify it; if the given allocation is not Pareto optimal, find a feasible Pareto improvement.

- (a)  $(x_A, y_A) = (12, 24)$  and  $(x_B, y_B) = (24, 24)$ .
- (b)  $(x_A, y_A) = (20, 20)$  and  $(x_B, y_B) = (20, 20)$ .
- (c)  $(x_A, y_A) = (20, 40)$  and  $(x_B, y_B) = (10, 10)$ .

4.2 The economy consists of two people (Mr. A and Mr. B) and two goods (the quantities of which will be denoted by  $x$  and  $y$ ). There is a single production process, which can turn the  $x$ -good into the  $y$ -good as follows:

The first four units of output can be produced at a (real) marginal cost of one-half input unit for each unit of output;  
the next four units at a marginal cost of one input unit for each unit of output;  
and remaining units at a marginal cost of two input units for each unit of output.

The total endowment is ten units of the input good (the  $x$ -good) and none of the output good (the  $y$ -good). Thus, the maximum output possible is ten units. Each consumer's preference is described by the utility function  $u(x, y) = xy$ .

Consider the following allocation: Each person consumes the bundle  $(x, y) = (2, 4)$ ; eight units of output are produced using six units of input. Is this allocation Pareto optimal? If so, prove it. If not, find a Pareto improvement.

4.3 A small town produces only a single product – apples – for sale in external markets. The town’s resources consist of two orchards (one containing only tall trees and the other containing only short trees) and two kinds of workers (giants and midgets). The technology for producing apples is such that one worker works with one tree, to produce apples according to the following table, which gives the daily apple production from each of the four possible ways that a worker can be combined with a tree:

	Tall Tree	Short Tree
Midget	1	3
Giant	8	4

There are 10 midgets and 20 giants in the town, and there are 40 tall trees and 5 short trees. None of the town’s resources can be used for any other purposes, either inside or outside the town.

(a) What is the efficient allocation of workers to trees? Are any of the resources unemployed in this allocation? Determine the marginal product of each of the four resources in this allocation.

Owners of the trees pay workers a piece rate – i.e., a per-apple wage. Each worker in the Tall Tree Orchard is paid  $P_T$  for each apple he picks, and each worker in the Short Tree Orchard is paid  $P_S$  per apple. The tree owners sell all apples that are picked; the apples are sold in the external apple market, where the price of an apple is  $P$ .

(b) If workers are free to move between orchards, what condition(s) must  $P_T$  and  $P_S$  satisfy in order to sustain the efficient allocation?

(c) Under competitive conditions, what will be the equilibrium piece rates and what profits (if any) will each of the resource owners earn? In equilibrium, determine whether any of the factor prices differ from the value of the factor’s marginal product.

(d) Now suppose it’s apple pickers who sell the apples in the external market. Each worker hires a tree, paying the tree’s owner for each apple the tree yields:  $R_T$  per apple to tall tree owners and  $R_S$  per apple to short tree owners. How will the competitive equilibrium differ from the one in (c)?

4.4 There are only two goods in the world, bread and wheat, quantities of which are denoted by  $x$  and  $y$ , respectively. There are 101 people in the economy, 100 of them called “consumers” and one “producer.”

Each consumer is endowed with 40 units of wheat and no bread and has a preference ordering described by the utility function  $u(x, y) = y - (1/2)x^2 + 8x$ .

The producer has no endowment of either good, but she is the sole owner of the economy’s only production process, which can turn wheat into bread at the rate of one bread unit for every four wheat units used as input. The producer cares only for wheat; i.e., her preference ordering is described by the utility function  $u(x, y) = y$ .

The consumers all behave as price-takers, but the producer behaves as a monopolist: the price of wheat is always \$1 per unit, and the producer sets the price  $p$  of her product (bread) so as to maximize her resulting utility (i.e., to maximize her dollar profit).

- (a) What price will the monopolist charge for each unit of bread, and how much will each consumer buy?
- (b) If the outcome in (a) is Pareto optimal, then verify it. If it’s not Pareto optimal, find a Pareto optimal allocation that makes all 101 people strictly better off than in (a).
- (c) Determine the consumer surplus, producer surplus, and total surplus in the monopoly outcome in (a) and at the Pareto optimal outcome you identified in (b).

4.5 Andy, Bob, and Cathy each have the same preferences for wine and grapes, described by the utility function  $u(x, y) = xy$ , where  $x$  and  $y$  denote an individual’s consumption of wine ( $x$  gallons) and grapes ( $y$  bushels). Grapes can be turned into wine; it takes three bushels of grapes to produce each gallon of wine. This production process is available to everyone – i.e., everyone has the ability to produce wine from grapes at this rate. Andy and Bob each own 12 bushels of grapes and Cathy owns 24 bushels of grapes. No one owns any wine. Determine the Walrasian equilibrium prices, production levels, and consumption bundles.

4.6 There are only two goods, grapes and wine. There is a single production process available, which can transform grapes into wine according to the following production function, in which  $z$  denotes the pounds of grapes used as input and  $f(z)$  the resulting quarts of wine obtained as output:

$$f(z) = \begin{cases} 0, & \text{if } 0 \leq z \leq 20 \\ z - 20, & \text{if } 20 \leq z \leq 80 \\ 20 + \frac{1}{2}z, & \text{if } z \geq 80. \end{cases}$$

There are ten identical consumers, each with a preference ordering described by the utility function  $u(x, y) = xy$ , where  $x$  and  $y$  denote pounds of grapes and quarts of wine consumed. Each consumer owns 12 pounds of grapes; there are no other grapes and there is no other wine at all except what is produced.

- (a) Draw the set of all the aggregate consumption bundles  $(x, y)$  that are feasible.
- (b) Determine all the Pareto optimal production-and-consumption plans in which each consumer receives the same bundle as every other consumer.
- (c) Consider the plan in which  $z = 60$ ;  $(x_{10}, y_{10}) = (24, 4)$ ; and  $(x_i, y_i) = (4, 4)$  for  $i = 1, \dots, 9$ . Find a Pareto improvement upon this plan.

For questions (d),(e), and (f), assume there is a single firm that owns the production process and the consumers all behave “competitively” – *i.e.*, they are price-takers. Assume that the consumers share equally in any profit that the firm earns.

- (d) Write down the firm’s demand correspondence for grapes, being careful to indicate the price lists for which the firm’s demand is not defined.
- (e) Assume that the price of grapes is three dollars per pound and the price of wine is five dollars per quart. How much will each consumer demand of each good?
- (f) Verify that there is no Walrasian equilibrium.

4.7 The only two goods in the economy are  $X$  and  $Y$ . Carol is the sole owner of the only firm in the economy, which can turn  $Y$  into  $X$  according to the production function  $q = f(z)$ , where  $f(z) = 2\sqrt{z}$ . Thus,  $z$  denotes the amount of  $Y$  used as input and  $q$  denotes the amount of  $X$  produced as output. Carol has no desire to consume any  $X$ ; she consumes only  $Y$ . There are two other people in the economy, Andy and Bert, whose preferences are described by the utility functions

$$u_A(x_A, y_A) = y_A + 12x_A - \frac{1}{2}x_A^2 \quad \text{and} \quad u_B(x_B, y_B) = y_B + 24x_B - \frac{1}{2}x_B^2,$$

where  $x_i$  and  $y_i$  denote individual  $i$ 's consumption of  $X$  and  $Y$ . The economy has no endowment of  $X$ , but each person owns 500 units of  $Y$ . Note that Andy's and Bert's marginal rates of substitution are  $MRS_A = 12 - x_A$  and  $MRS_B = 24 - x_B$ , and that the real marginal cost of production is  $(1/2)q$ .

- (a) Verify that there is a Pareto efficient production-and-consumption plan in which  $q = 18$ .
- (b) Are there any other Pareto efficient production-and-consumption plans? If so, find one. If not, verify that there aren't.
- (c) Find a Walrasian equilibrium (i.e., a market equilibrium) in which the firm and all three consumers are price-takers. (Assume that the price of  $Y$  is one dollar per unit.) Determine the price of  $X$ , the output and input levels, the profit (if any), and the bundles that all three people consume.
- (d) If Carol operates the firm as a monopoly she will charge a price of \$12 for each unit of  $X$  she sells. Verify that price, and determine how much she will produce, her profit, and the resulting consumption bundles.
- (e) Find a plan that makes everyone strictly better off than in (d), i.e., a strict Pareto improvement. (The Pareto improvement you find need not be Pareto efficient.)
- (f) Find a Pareto efficient plan that makes everyone strictly better off than in (d).

4.8 Goods X and Y are jointly produced, with labor as the only input. The price of labor is one dollar per unit hired. In order to produce  $x$  units of X and  $y$  units of Y, the firm must hire  $\max(x, y)$  units of labor. The market demand for the two goods is given by the functions  $x = 2 - p_x$  and  $y = 3 - p_y$ .

- (a) Determine the competitive equilibrium production levels and prices of X and Y.
- (b) Determine the production levels and prices if the goods are produced by a single firm that has a monopoly in each of the two markets.

4.9 The only two goods in the economy are X and Y. There is only one firm in the economy, which can turn X into Y according to the production function  $q = f(z)$ , where  $f(z) = 20\sqrt{z}$ . Thus,  $z$  denotes the amount of X used as input, and  $Q$  denotes the amount of Y produced as output. There are two consumers, Amy and Bev, whose preferences are described by the utility functions

$$u_A(x_A, y_A) = 2x_A + y_A \quad \text{and} \quad u_B(x_B, y_B) = \frac{1}{2}x_B + y_B,$$

where  $x_i$  and  $y_i$  denote individual  $i$ 's consumption of X and Y. The economy has no endowment of Y, but each consumer owns 200 units of X. Amy owns the firm: she receives whatever profits it earns.

Determine all Walrasian equilibria. (Note that all consumers and firms are assumed to be price-takers in a Walrasian equilibrium, no matter how few of them there are.)

4.10 The tiny country of Dogpatch has 90 residents and 10 identical machines. Ten of the people (called “capitalists”) own one machine apiece, but are unable to provide any useful labor services. Each of the remaining 80 people (called “workers”) has the capacity to work with machines and the other workers to produce shmoos, but none of the workers owns any machines. Combining  $x$  workers with  $y$  machines yields  $F(x, y)$  shmoos, where

$$F(x, y) = x^{\frac{2}{3}}y^{\frac{1}{3}} \quad \text{for all } (x, y) \in \mathbb{R}_+^2.$$

It is possible to divide a worker’s time among any number of machines and to divide a machine’s time among any number of workers.

No one in Dogpatch cares about consuming shmoos, but shmoos can be sold in the neighboring country of Alcappia for a dollar apiece. Everyone in Dogpatch uses dollars to purchase in Alcappia the goods that he does care about consuming. All residents of Dogpatch are price-takers in all markets, and everyone understands how to use the constant-returns-to-scale technology embodied in the function  $F$ . Machines and labor are neither imported nor exported by Dogpatch.

(a) Suppose that the capitalists are the entrepreneurs: each one hires workers and combines them with his machine, and then sells the resulting production of shmoos. What will be the equilibrium wage rate and total production of shmoos, and how many dollars will each capitalist and each worker spend in Alcappia?

(b) Now suppose the workers are the entrepreneurs: each worker rents machine time, which he combines with his own labor, and then sells the resulting production of shmoos. What will be the equilibrium rental price and total production of shmoos, and how many dollars will each capitalist and each worker spend in Alcappia?

(c) Is either of the institutional arrangements in (a) or (b) Pareto optimal for the residents of Dogpatch? If so, explain why; if not, find a Pareto improvement.

4.11 Alice owns no simoleons “today,” but she will own 30 simoleons “tomorrow.” Her preference for alternative consumption streams is described by the utility function  $u_A(x, y) = \min\{x, y\}$ , where  $x$  and  $y$  denote the number of simoleons she consumes today ( $x$ ) and tomorrow ( $y$ ). Betsy owns 20 simoleons today, but she will own none tomorrow; her preference is described by the utility function  $u_B(x, y) = xy$ .

(a) If Alice and Betsy engage in a borrowing-and-lending market (in which they’re the only participants) in order to arrive at more desirable consumption streams than they’re endowed with, and if each behaves “competitively” (taking the interest rate as given), what will be the equilibrium rate of interest and the equilibrium consumption stream of each?

(b) Verify that Walras’ Law is satisfied at the equilibrium rate of interest. Is Walras’ Law satisfied at any non-equilibrium interest rates, and if so, at which ones? Verify your answer.

(c) What is the net present value (NPV) of each woman’s wealth (i.e., of her endowment stream) at the equilibrium rate of interest?

(d) Now suppose the women’s endowment streams are (15,15) for Alice and (5,15) for Betsy. Is this situation Pareto optimal? Verify your answer.

(e) Is there an interest rate at which the endowment streams in (d) are a Walrasian equilibrium (i.e., an interest rate at which, if the women are endowed with intertemporal allocation ((15,15) (5,15)), there will be no intertemporal trade)? Are there any other endowment streams for which this allocation — i.e., (15,15) to Alice and (5,15) to Beth — is a Walrasian equilibrium? If so, determine a necessary condition on each woman’s wealth (the NPV of her endowment stream) that must be satisfied if the allocation ((15,15) (5,15)) is a Walrasian equilibrium.

For parts (f) and (g), assume that a single real investment process exists by which sacrificing simoleons today to be used as input will yield output of three times as many simoleons tomorrow.

(f) Assume that the women’s endowment streams are (20,0) for Alice and (10,0) for Betsy. Determine all Walrasian equilibrium interest rates, production plans, and consumption plans for each woman. What is the aggregate amount of profit? Does it matter who owns the investment (production) process? Why, or why not?

(g) Consider the plan in which 15 simoleons are used today as input to the investment process, Alice’s consumption stream is (5,5), and Betsy’s is (10,40). Find a Pareto improvement upon this plan (but not necessarily a Pareto optimal one) in which *both* women are strictly better off.

4.12 Jerry is shipwrecked on a tropical island. Fortunately, the island has a small grove of orange trees, and while Jerry doesn't care for oranges, he does like orange juice. Even *more* fortunately, he saved two orange juice machines before his ship went down. Jerry refers to the machines as Firm 1 and Firm 2. Each machine is capable of producing orange juice from oranges, but one machine is more efficient than the other. Specifically, the machines turn  $z$  oranges per day into  $q$  ounces of orange juice per day according to the production functions

$$q_1 = f_1(z_1) = 12(\sqrt{z_1 + 1} - 1) \quad \text{and} \quad q_2 = f_2(z_2) = 12(\sqrt{z_2 + 4} - 2).$$

Note that the machines' marginal productivities are

$$f'_1(z_1) = 6/\sqrt{z_1 + 1} \quad \text{and} \quad f'_2(z_2) = 6/\sqrt{z_2 + 4}.$$

Firm 1 is uniformly more efficient than Firm 2: for any input  $z$  of oranges,  $f_1(z) > f_2(z)$ ; moreover, Firm 1's *marginal* productivity is also larger at any level of operation:  $f'_1(z) > f'_2(z)$ . Should Jerry therefore use Firm 1 exclusively? Let's find out.

(a) Suppose Jerry's orange grove yields 13 oranges each day. He simply wants to use the 13 oranges as inputs into his machines every day in whatever way will produce the most orange juice. Determine the greatest amount of orange juice Jerry can produce each day, and determine how many oranges he must put into each machine each day in order to obtain that maximum level of production.

(b) Instead of 13 oranges per day, suppose Jerry's orange grove yields only 3 oranges per day. How many oranges should he put into each machine and how much orange juice will he produce? What if his orange grove yields *fewer* than 3 oranges per day?

Jerry decides that he likes oranges after all. His preferences for oranges and orange juice are described by the utility function

$$u(x, y) = x + 2y - \frac{1}{48}y^2,$$

where  $x$  and  $y$  denote, respectively, oranges consumed per day and ounces of orange juice consumed per day. Jerry's orange grove is now producing 30 oranges per day.

(c) How many oranges per day will Jerry consume and how many will he put into each machine? How much orange juice will he produce and consume? Suppose he wants to decentralize this plan; what are the decentralizing (*i.e.*, efficiency) prices he will need to use? How much profit should he attribute to each "firm?"

Jerry discovers a second survivor of the shipwreck — Kramer! The two of them agree to divide up ownership of the island’s resources. Kramer assumes 100% ownership of the eastern 1/3 of the orange grove (yielding 10 oranges per day) and 100% ownership of the more efficient firm, Firm 1. Jerry assumes 100% ownership of the less efficient Firm 2 and, to compensate for getting the less efficient firm, he receives 100% ownership of the western 2/3 of the orange grove (yielding 20 oranges per day). Jerry’s and Kramer’s preferences are described by the utility functions

$$u_J(x, y) = x + 2y - \frac{1}{40}y^2 \quad \text{and} \quad u_K(x, y) = x + y - \frac{1}{24}y^2$$

(d) If Jerry and Kramer each behave as price-takers in their consumption and production decisions, and each one operates his firm so as to maximize its profit, what will be the equilibrium prices? How much will each one consume of each good, how many oranges will each firm purchase and use as inputs, how much orange juice will each firm produce, and how much profit will each firm earn? Will the outcome be Pareto efficient?

(e) How would your answers to (d) change if ownership of the firms were different? For example, what if ownership were reversed, Jerry owning Firm 1 and Kramer Firm 2? What if each one owns 50% of each firm?

(e) Determine Jerry’s and Kramer’s consumer surplus and the firms’ producer surplus. Is the total surplus a good measure of an outcome’s welfare?

4.13 As in Harberger’s classic example [JPE 1962], assume that there are two goods produced: product  $X$  is produced by firms in the “corporate” sector and product  $Y$  by firms in the “non-corporate” sector. Both products are produced using the two inputs **labor** and **capital** (quantities denoted by  $L$  and  $K$ ). Production functions are

$$X = \sqrt{L_X K_X} \quad \text{and} \quad Y = \sqrt{L_Y K_Y}.$$

All consumers have preferences described by the utility function  $u(X, Y) = XY$ . The consumers care only about consuming  $X$  and  $Y$ , and they supply labor and capital inelastically in the total amounts  $L = 600$  and  $K = 600$ . Let  $p_X$ ,  $p_Y$ ,  $p_L$ , and  $p_K$  denote the prices of the four goods in the economy; assume that  $p_L = 1$  always.

(a) What is the Walrasian equilibrium?

(b) Suppose that a 50% tax is imposed on payments to capital in the corporate sector only, and that the government uses the tax proceeds to purchase equal amounts of the output of the two sectors. What will be the new Walrasian equilibrium? How is welfare affected by the tax – are people better off with or without the tax?

4.14 Benjamin has just graduated from college. Now he's deciding how much further to invest in his own human capital. The result of his decision will be a consumption stream  $\mathbf{x} = (x_0, x_1)$ , where  $x_0$  denotes consumption "today" (say, during the next ten years) and  $x_1$  denotes consumption "tomorrow" (the remainder of his life). His preference is described by the utility function  $u(x_0, x_1) = x_0^3 x_1^2$ . If he undertakes no investment, and neither saves nor borrows, his consumption stream will be  $\hat{\mathbf{x}} = (\hat{x}_0, \hat{x}_1) = (38, 16)$ .

The investment possibilities available to Benjamin are described by the function

$$f(z) = \begin{cases} 4z - \frac{1}{8}z^2, & z \leq 16 \\ 32, & z \geq 16. \end{cases}$$

(a) Suppose Benjamin has no access to capital markets: he can neither borrow nor save. Verify that he will invest at level  $z = 8$ . What will be his resulting consumption stream, marginal rate of substitution, and marginal rate of transformation? Depict this decision in a diagram showing Benjamin's consumption-possibilities set and the best indifference curve he can attain.

Now suppose Benjamin has access to capital markets in which he can borrow and lend (i.e., save) at a common inter-period (not annual) interest rate,  $r$  – i.e., if he borrows  $B$  dollars today, he repays  $(1 + r)B$  dollars tomorrow; if he saves  $S$  today, he receives  $(1 + r)S$  tomorrow.

(b) If the interest rate is 100%, how much will Benjamin invest? How much will he borrow or lend? What will be his resulting consumption stream? Verify that the net present value of his consumption stream is equal to the net present value of his endowment stream plus the net present value of his investment plan.

(c) Now suppose the interest rate is 200%, and consider the investment and consumption plans in part (b). Verify that the present value of the consumption stream in (b) is still equal to the present value of Benjamin's endowment stream plus the present value of the investment plan in (b). In a diagram like the one in part (a), depict the present-value contour(s) on which the investment plan and consumption plan lie. Will Benjamin's investment and consumption plans be the same as in (b)? If not, will his investment, borrowing, lending, and consumption in each period be more or less than in (b)? Hint: If  $r = 200\%$ , the present value of the investment plan in part (b) is zero.

(d) If the interest rate is 200%, how much will Benjamin invest? How much will he borrow or lend? What will be his resulting consumption stream? Hint: Benjamin will choose a consumption plan that maximizes his utility among all plans with a net present value no greater than his wealth. His wealth is the present value of his endowment plus the net present value of his investment plan. You should find that his wealth is 44.

## The Core: Bargaining Equilibrium

5.1 There are three consumers; each one's preference is represented by the utility function  $u(x, y) = xy$ . The first consumer owns the bundle (19,1), the second owns the bundle (1,19), and the third owns the bundle (10,10). Both goods are divisible. Determine the set of all core allocations.

5.2 Bart owns the bundle (16,4); Lisa owns the bundle (8,8); Krusty owns the bundle (4,16). Each one's preference is described by the utility function  $u(x, y) = xy$ . Consider the proposed allocation in which Bart gets the bundle (10,7); Lisa gets the bundle (7,10); and Krusty gets the bundle (4,16). Determine whether the coalition consisting of Bart and Lisa can improve upon the proposed allocation.

5.3 There are two goods and three people in the economy, and all three people have the same utility function:  $u(x, y) = xy$ . Person #1 is endowed with the bundle (12, 0), and Persons #2 and #3 are each endowed with the bundle (0, 12). In the following cases determine whether the given allocation is in the economy's core. If it is, verify that it is; and if it's not, find an allocation with which some coalition can unilaterally make each of its members better off.

- (a)  $(x_1, y_1) = (8, 8), \quad (x_2, y_2) = (2, 8), \quad (x_3, y_3) = (2, 8)$
- (b)  $(x_1, y_1) = (4, 8), \quad (x_2, y_2) = (4, 8), \quad (x_3, y_3) = (4, 8)$
- (c)  $(x_1, y_1) = (7, 14), \quad (x_2, y_2) = (3, 6), \quad (x_3, y_3) = (2, 4)$

5.4 There are four consumers, each one's preference is represented by the utility function  $u(x, y) = xy$ . Two consumers (say, #1 and #3) own all of one good – each owns one unit – and the other two consumers own all of the other good – again, each owns one unit. Both goods are fully divisible. Determine the set of all core allocations.

5.5 Construct an example to show that if the consumer types are not present in equal numbers, then it is not necessarily true that identical consumers are treated equally in core allocations.

5.6 There are two goods (quantities are denoted by  $x$  and  $y$ ) and no production is possible. Amy, Bev, and Cal all have the exact same preferences for the goods, represented by the utility function  $u(x, y) = xy$ . Amy owns the bundle  $(12, 4)$ , Bev owns the bundle  $(4, 4)$ , and Cal owns the bundle  $(4, 12)$ . Determine whether each of the following allocations is in the core, and show why your answer is the right one.

- (a)  $(x_A, y_A) = (8, 8), \quad (x_B, y_B) = (4, 4), \quad (x_C, y_C) = (8, 8)$   
 (b)  $(x_A, y_A) = (4, 8), \quad (x_B, y_B) = (4, 4), \quad (x_C, y_C) = (7, 7)$

5.7 Amy and Bev each own four loaves of bread and no honey. Cal owns eight pounds of honey, but no bread. All three have preferences described by the utility function  $u(x, y) = xy$ , where  $x$  denotes the loaves of bread consumed and  $y$  denotes pounds of honey. Determine whether the following allocations are in the core:

- (a) Amy:  $(1, 1)$  Bev:  $(3, 3)$  Cal:  $(4, 4)$   
 (b) Amy:  $(2, 2)$  Bev:  $(2, 2)$  Cal:  $(4, 4)$

5.8 Jerry and Elaine have each ordered a large pizza (12 slices each), but have found they have nothing to drink with their pizzas. Kramer has two six-packs of beer (12 bottles), but nothing to eat. They decide to get together for dinner. Each has the same preferences, described by the utility function  $u(x, y) = xy$ , where  $x$  and  $y$  denote slices of pizza and bottles of beer.

- (a) Derive the utility frontier for each coalition.  
 (b) Determine whether the following allocation is in the core:

$$(x_J, y_J) = (6, 3) \quad (x_E, y_E) = (4, 2) \quad (x_K, y_K) = (14, 7).$$

- (c) Kramer is studying economics and recalls that core allocations treat identical individuals identically. What does this “theorem” tell you about the answer to (b)?

5.9 (See Exercises 1.6 and 3.2) The Arrow and Debreu families live next door to one another. Each family has an orange grove that yields 30 oranges per week, and the Arrows also have an apple orchard that yields 30 apples per week. The two households' preferences for oranges ( $x$  per week) and apples ( $y$  per week) are given by the utility functions

$$u_A(x_A, y_A) = x_A y_A^3 \quad \text{and} \quad u_D(x_D, y_D) = 2x_D + y_D.$$

The Arrows and Debreus realize they may be able to make both households better off by trading apples for oranges.

- (a) Determine all the Pareto efficient allocations and depict them in an Edgeworth box diagram.
- (b) Determine all Walrasian equilibrium price lists and allocations.
- (c) Determine all the core allocations.

5.10 Amy has six bottles of beer and Beth has eight bags of peanuts. Amy has no peanuts and Beth has no beer. Amy's and Beth's preferences for beer and peanuts are described by the utility functions

$$u^A(x, y) = y + 12x - x^2 \quad \text{and} \quad u^B(x, y) = y + 12x - \frac{1}{2}x^2,$$

where  $x$  denotes bottles of beer consumed and  $y$  denotes bags of peanuts consumed. Beer and peanuts are the only goods we will consider, and Amy and Beth are the only traders. Let  $Z$  denote the allocation in which Amy consumes  $(x, y) = (4, 8)$  and Beth consumes  $(x, y) = (2, 0)$ .

- (a) Is  $Z$  Pareto efficient?
- (b) Draw an Edgeworth box depicting the set of all Pareto optimal allocations.
- (c) Is  $Z$  in the core?
- (d) Is it a Walrasian equilibrium allocations? If so, give an equilibrium price-list.
- (e) Is the Walrasian equilibrium price ratio unique?

5.11 Each person in the following questions cares only about the amounts of the two goods that are allocated to her, and not about how much is allocated to others. Each one's preference is described by the utility function  $u(xy) = xy$ .

- (a) Abby owns the bundle (3,1), Beth owns the bundle (1,3). Determine all the core allocations.
- (b) Now suppose that Abby and Beth are joined by a third person, Cathy, who has the same preference as the others, but who owns the bundle (1,1). Determine all the competitive (*i.e.*, Walrasian) equilibria for this three-person economy.
- (c) In the three-person economy of (b), determine whether the coalition consisting of Abby and Cathy can unilaterally improve upon the proposed allocation in which Abby and Beth each receive the bundle (2,2) and Cathy receives the bundle (1,1). Prove that your answer is the correct one.
- (d) Now suppose the economy consists of 200 people, all of whom have the same preference, described by the utility function  $u(x, y) = xy$ , and that half the people each own the bundle (3,1) and the other half each own the bundle (1,3). Give as complete a description of the core allocations as you can.

5.12 Ann owns 12 bags of peanuts, but no beer. Bill owns 6 bottles of beer, but no peanuts. Ann and Bob have identical preferences, given by the utility function  $u(x, y) = x^2y$ , where  $x$  and  $y$  denote bags of peanuts and bottles of beer consumed, respectively. The Walrasian equilibrium allocation is the one in which Ann consumes the bundle  $(\hat{x}_A, \hat{y}_A) = (8, 4)$  and Bob consumes the bundle  $(\hat{x}_B, \hat{y}_B) = (4, 2)$ . Now suppose that Ann and Bob are joined by Amy and Bill. Amy is identical to Ann (same endowment and same preferences) and Bill is identical to Bob. Note that you don't need to calculate the utility frontiers to solve this problem; those calculations are a bit complicated.

- (a) Now that all four people are available to trade with one another, the only Walrasian allocation is the one that gives *both* Ann and Amy the bundle  $(\hat{x}_A, \hat{y}_A)$  and both Bob and Bill the bundle  $(\hat{x}_B, \hat{y}_B)$ . Prove that this allocation is in the core.
- (b) Now consider the bundles  $(\bar{x}_A, \bar{y}_A) = (4, 2)$  and  $(\bar{x}_B, \bar{y}_B) = (8, 4)$ . When Ann and Bob are the only two people who are going to exchange beer for peanuts, it is easy to see that this allocation is in the core. Determine whether the allocation that gives the bundle  $(\bar{x}_A, \bar{y}_A)$  to both Ann and Amy, and the bundle  $(\bar{x}_B, \bar{y}_B)$  to both Bob and Bill, is a core allocation when all four of them can exchange beer and peanuts with one another.

5.13 Amy, Beth, and Carol each own orange trees, and they each like to eat oranges and drink orange juice. Their preferences and their trees' weekly yield of oranges are as follows, where  $x_i$  denotes  $i$ 's consumption of oranges per week and  $y_i$  denotes  $i$ 's spending (in pennies per week) on all other goods:

	Yield	Utility Function
Amy	10	$y_A + 60x_A - 3x_A^2$
Beth	15	$y_B + 60x_B - (3/2)x_B^2$
Carol	5	$y_C + 60x_C - x_C^2$

Each woman has been consuming only the oranges from her own tree. Find a Pareto improvement that is also in the core. Explain how you know that your proposed allocation is both a Pareto improvement and in the core.

5.14 (See Exercise 4.5) Andy, Bob, and Cathy each have the same preferences for wine and grapes, described by the utility function  $u(x, y) = xy$ , where  $x$  and  $y$  denote an individual's consumption of wine ( $x$  gallons) and grapes ( $y$  bushels). Grapes can be turned into wine; it takes three bushels of grapes to produce each gallon of wine. This production process is available to everyone – i.e., everyone has the ability to produce wine from grapes at this rate.

(a) Suppose Andy and Bob each own 12 bushels of grapes and Cathy owns 24 bushels of grapes. No one owns any wine. Determine the Walrasian equilibrium prices, production levels, and consumption bundles.

In parts (b) and (c), either prove that the proposed allocation is in the core (by showing that no coalition can do better for itself), or prove that it is not (by showing that some coalition can do better for itself).

(b) With the endowments in (a), determine whether the following allocation is in the core:

$$\text{Andy: } (1,3) \quad \text{Bob: } (3,9) \quad \text{Cathy: } (4,12)$$

(c) Now suppose that Andy and Bob each own 4 gallons of wine, but no grapes, and Cathy owns 24 bushels of grapes, but no wine. (It is not possible, of course, to turn wine into grapes.) Determine whether the following allocation is in the core:

$$\text{Andy: } (2,6) \quad \text{Bob: } (4,0) \quad \text{Cathy: } (4,12)$$

5.15 Acca and Bayab are tiny islands in the gulf of Ababa. Anna is the sole resident and owner of Acca, and Bob is the sole resident and owner of Bayab. Anna and Bob consume only apples and bananas, and each has the same preference ordering, represented by the utility function  $u(x, y) = xy$ , where  $x$  and  $y$  denote the consumer's daily consumption of apples and bananas. Only apples are grown on Acca, where the yield is 20 apples per day, and only bananas are grown on Bayab, where the yield is 10 bananas per day.

Anna and Bob have been trading with one another (by boat) for years: Anna gives Bob 12 apples every day in return for 4 bananas. Call this situation and the resulting consumption allocation  $S$ , for "status quo."

- (a) Is  $S$  Pareto Optimal? Is it in the core? Verify your answers.
- (b) Is  $S$  a Walrasian equilibrium allocation? If so, determine the associated prices; if not, could both Anna and Bob be made better off by organizing their exchanges in terms of markets and prices?

Now suppose that Anna discovers a technology, called the Alpha technology, with which she can transform apples into bananas, at the rate of two apples for each banana obtained. Answer (c), (d), and (e) assuming that Anna is the sole owner of the Alpha technology.

- (c) Now is  $S$  Pareto optimal? If so, verify it; if not, find a Pareto improvement
- (d) How does the discovery of the new technology affect the set of core allocations? Is  $S$  in the core now?
- (e) How does the discovery of the new technology affect the set of Walrasian allocations and prices?
- (f) Now suppose it is not Anna who discovers a technology, but it is Bob. Bob discovers the Beta technology, with which he can transform two bananas into one apple as often as he likes. Then is  $S$  in the core? Will the set of Walrasian allocations and/or prices be affected (as compared to the original, no-technology situation)?

(g) Now suppose that both technologies have been discovered. How will the set of Walrasian allocations and prices be affected by the ownership of the technologies? In particular, compare the Walrasian equilibria when Anna owns Alpha and Bob owns Beta to when Anna owns Beta and Bob owns Alpha. What would happen to the Walrasian outcomes if one of the people owned both technologies?

5.16 (See Exercises 1.2 and 3.6) Ann and Bob each own 10 bottles of beer. Ann owns 20 bags of peanuts and Bob owns no peanuts. There are no other people and no other goods in the economy, and no production of either good is possible. Using  $x$  to denote bottles of beer and  $y$  to denote bags of peanuts, Ann's and Bob's preferences are described by the following utility functions:

$$u_A(x_A, y_A) = x_A y_A^4 \quad \text{and} \quad u_B(x_B, y_B) = 2x_B + y_B.$$

Note that their *MRS* schedules are  $MRS_A = y_A/4x_A$  and  $MRS_B = 2$ .

- (a) Determine all Walrasian equilibrium price lists and allocations.
- (b) Determine all core allocations.

## Imperfect Competition

6.1 There are only two firms producing a particular product. The demand for the product is given by the relation  $p = 24 - Q$ , where  $p$  denotes the price (in dollars per unit) and  $Q$  denotes the total quantity sold by the two firms. The firms both have constant marginal cost: it costs Firm #1 eight dollars for each unit it produces and Firm #2 four dollars for each unit it produces. Neither firm has any fixed costs.

(a) Assume that each firm behaves as if its own decisions will not affect the quantity that the other firm tries to sell, and that each firm tries to maximize its own profit – in other words, each behaves as a Cournot duopolist. Determine the equilibrium price and quantity in the market, and determine each firm's production and profit.

(b) Determine the same items as in (a) under the assumption that the firms cooperate fully with one another.

(c) Determine the same items as in (a) under the assumption that each firm behaves as a price taker, taking the market price as given when making its decision.

6.2 There are only two limousine firms capable of driving passengers between the airport and downtown. The two firms' services are identical (in particular, each carries only a single passenger on each trip), but the firms' costs of production differ: it costs one of the firms only \$10 per trip and it costs the other \$20 per trip. The market demand for limousine trips from the airport to downtown is given by the equation  $Q = 240 - 4p$ , where  $Q$  denotes the number of trips purchased per week when the price is  $p$  dollars per trip.

(a) What is the Cournot equilibrium in this market?

(b) If the firms cooperate fully to maximize their joint profits, how many trips will each firm make?

(c) There are forty consumers. Each one has a weekly income of at least \$300 and each one's preference is described by the utility function  $u(x, y) = y - (1/20)(60 - 10x)^2$ , where  $x$  is the number of trips she makes per week and  $y$  is the number of dollars she has available to spend on other goods. Each firm cares only about its own profits. Find an allocation that makes all the consumers *and* each of the two firms strictly better off than in the Cournot equilibrium allocation.

(d) Determine the consumer surplus, producer surplus, and total surplus at each of the outcomes in (a), (b), and (c).

6.3 There are only two goods in the world, bread and wheat, and there are 102 people, 100 of them called “consumers” and two called “producers.” Each consumer is endowed with 140 pounds of wheat and no bread and has preferences described by the utility function  $u(x, y) = y - (1/2)x^2 + 20x$ , where  $x$  and  $y$  are the number of loaves of bread and pounds of wheat that he consumes. The consumers all behave as price-takers, and the price of wheat is always \$1 per unit. Thus, each consumer’s demand function for bread is  $x = 20 - p$ , where  $p$  is the price of bread (in dollars per loaf).

Each producer has no endowment of either good, but owns a machine that can turn wheat into bread, producing a loaf of bread for every eight pounds of wheat used as input. There is no other way for anyone in the economy to transform wheat into bread or bread into wheat. Each producer cares only for wheat; i.e., his preferences are described by  $u(x, y) = y$ .

(a) Suppose the two producers behave as Cournot duopolists. Determine the equilibrium price of bread, the amount produced by each producer, and the resulting consumption of bread and wheat by each of the economy’s 102 participants.

(b) If the allocation in (a) is Pareto optimal, verify that it is. If it is not, find a Pareto optimal allocation in which all 102 members of the economy are strictly better off.

(c) What if each of the consumers were endowed with only 120 pounds of wheat?

6.4 There are only two firms producing a particular product. Demand for the product is given by the equation  $Q = 24 - p$ , where  $p$  denotes the price at which the product is sold (in dollars) and  $Q$  denotes the resulting quantity demanded. The two firms’ products are perfect substitutes to all consumers, so both firms receive the same price for every unit sold. Firm 1’s cost of production is \$15 per unit and Firm 2’s is \$18 per unit; neither firm incurs any fixed costs.

(a) Determine the market price and each firm’s production under each of the following assumptions:

(a’) The market is competitive.

(a’’) The firms behave as Cournot duopolists. Include a diagram of the two firms’ reaction curves.

(a’’’) The firms collude. Include bounds on the monetary transfers between the firms.

(b) Now assume that Firm #1’s unit cost has fallen to \$6. Draw the firms’ reaction curves, and compare the market outcome under Cournot behavior with the outcome under collusive behavior.

(c) Compare the core in (a) and (b), assuming the only players are the two firms.

6.5 There are three firms selling a homogeneous good. Demand for the good is given by  $p = 300 - Q$ , where  $Q$  denotes the total quantity sold by all three firms. The firms all have constant per-unit costs of production: Firm 1's cost is \$20 per unit of output, Firm 2's is \$40 per unit, and Firm 3's \$80 per unit. Determine the Cournot equilibrium.

6.6 Firm 1 and Firm 2 are the only firms that can produce in a particular market. The firms' costs of production are described by the cost functions

$$C_1 = 4x_1 \quad \text{and} \quad C_2 = K + 2x_2.$$

for positive levels of output  $x_1$  and  $x_2$ . Each firm's cost is zero if it produces at level zero. The market demand function for the (homogeneous) good produced by the firms is  $Q = 12 - p$ , where  $p$  is the price and  $Q$  is the quantity demanded.

(a) If the firms cooperate with one another, operating as a cartel, determine the levels of output  $x_1$  and  $x_2$  by each firm and the market price  $p$ , all as a function of Firm 2's fixed cost  $K$ .

(b) Now suppose that the two firms do not cooperate: each firm attempts to maximize its own profit, and each assumes that its own actions will have no effect upon the quantity that the other will offer for sale in the market. In a single diagram, draw each firm's reaction curve. What will be the Nash equilibrium outcome (each firm's output and the market price) if  $K = 16$ ? How would your answer change (if at all) if  $K > 16$ ?

(c) Now suppose that the two firms behave "competitively" — *i.e.*, each takes the market price as given and chooses an output level that will maximize its profit. What will be the equilibrium outcome?

6.7 There are only two firms in a market in which the demand curve for the product the firms produce is  $Q_D = 24 - p$ , where  $p$  is the market price of the product in dollars and  $Q_D$  is the quantity demanded. It costs Firm 1 six dollars for each unit it produces, and it costs Firm 2 eight dollars for each unit it produces. Neither firm has any fixed costs. Each firm has a capacity constraint: Firm 1 can produce no more than six units, and Firm 2 can produce no more than eighteen units.

- (a) Determine the outcome (price, production levels, total profit) if the two firms cooperate to maximize joint profits. Draw the total and marginal cost curves for the joint-profit maximization problem
- (b) Determine the outcome if each firm is a price-taker. Draw the market demand and supply curves.
- (c) Determine the outcome if the firms behave as Cournot duopolists. Draw the firms' reaction functions.

6.8 There are two firms supplying a particular market. Consumers do not regard the two firms' products as perfect substitutes for one another: if the two firms charge the respective prices  $p_1$  and  $p_2$  for their products, then sales will be  $q_1 = 12 - .3p_1 + .2p_2$  and  $q_2 = 36 + .1p_1 - .4p_2$ . The firms' cost functions are  $C_1(q_1) = 10q_1$  and  $C_2(q_2) = 20q_2$ .

- (a) Determine the Bertrand equilibrium
- (b) Determine the Cournot equilibrium
- (c) Determine the outcome if the two firms fully cooperate to maximize their joint profits.

*Note:* the Bertrand and cooperative outcomes are not nice integers. The Bertrand equilibrium is  $(p_1, p_2) = (\$45.22, \$60.65)$ , approximately; and the cooperative outcome is  $(q_1, q_2) = (9.23, 12.05)$ , approximately.

The fact that the Cournot and Bertrand outcomes differ might seem a little bit puzzling: if I take my rival's action as given, this leaves me facing a downward-sloping demand curve for my product, so it will not matter whether my "decision variable" is my price or my quantity, because one determines the other via the demand curve. What is the explanation of this seeming paradox?

6.9 Airhead and Bubbles are the only two firms producing natural spring water. The firms draw their waters from different springs (Airhead’s water is “still” and Bubbles’ is carbonated), so each firm has some “market power.” Specifically, the demands for the two firms’ waters are given by

$$q_A = 30 - 2p_A + p_B \quad \text{and} \quad q_B = 15 - 2p_B + p_A,$$

where  $p_i$  denotes the price per gallon charged by Firm  $i$  and  $q_i$  denotes the resulting number of gallons the customers of Firm  $i$  will purchase. Each firm’s production is costless.

(a) Determine the equilibrium prices and quantities if each firm takes as given the price charged by the other firm (*i.e.*, find the Bertrand equilibrium). Draw the two firms’ reaction curves in a single diagram.

(b) The Bertrand equilibrium is often said to be the outcome if the firms “compete in prices” and the Cournot equilibrium the outcome if the firms “compete in quantities.” But suppose one of the firms – say, Airhead – takes its rival’s price as given. This leaves Airhead facing a specific downward-sloping demand curve, and it makes no difference whether Airhead “chooses price” or “chooses quantity”: choosing either one determines the other. In other words, it seems to make no difference whether the firms “choose prices” or “choose quantities” – the outcome ought to be the same. However, if you calculate the Cournot equilibrium for Airhead and Bubbles, you’ll find that it’s different than the Bertrand equilibrium: different prices and different quantities. What is the explanation of this seeming paradox?

6.10 Consider a market in which all buyers are price-takers, each with the demand function  $x = 40 - p$ , where  $p$  denotes the price of the product and  $x$  the quantity the consumer purchases. There are two firms supplying this market, and the two firms are located near one another. Production generates air pollution, and the pollution in turn increases each firm's cost of production. Specifically, every unit produced by either firm adds one unit of pollution to the air, and the firms' cost functions are  $C_1(q_1) = 2Aq_1$  and  $C_2(q_2) = 3Aq_2$ , where  $A$  is the total amount of pollution in the air, and where  $q_i$  is the per-capita production by Firm  $i$  (*i.e.*, the firm's production divided by the number of buyers in the market). The pollution level  $A$  is equal to the total per-capita production by the two firms,  $q_1 + q_2$ . Assume throughout that each firm understands how its own marginal cost is affected the amount of pollution and thus by the firms' production levels.

Determine the market price and each firm's production level under each of the following behavioral assumptions:

- (a) Each firm takes the other's production level as parametric and maximizes profit.
- (b) The two firms cooperate fully as a cartel, maximizing joint profits.
- (c) Each firm behaves competitively — *i.e.*, is a price-taker — and maximizes profit.

6.11 Two firms produce similar but differentiated products. They choose their prices strategically, each firm taking the other's price as given, *i.e.*, unaffected by its own decisions. The demand for each firm's product and its cost are given by the equations

	Demand	Costs
Firm 1:	$q_1 = 120 - 30p_1 + 20p_2$	$C_1(q_1) = 4q_1$
Firm 2:	$q_2 = 240 + 10p_1 - 20p_2$	$C_2(q_2) = 8q_2$

(a) Determine the Bertrand equilibrium prices, output levels, and profits.

Inverting the demand functions given above, we obtain the following (equivalent) description of the demand for the firms' products:

Firm 1:	$p_1 = 18 - (1/20)q_1 - (1/20)q_2$
Firm 2:	$p_2 = 21 - (1/40)q_1 - (3/40)q_2$

(b) Determine the Cournot equilibrium output levels, prices, and profits. You should find that they're different than the ones you obtained in (a).

(c) Using the first demand system and treating the prices as the decision variables, determine the prices, output levels, and profits if the firms collude to maximize their total profit. Then do the same, using the second demand system and treating the quantities as the decision variables. You should obtain the same outcome both ways.

(d) Suppose Firm 2 is charging the price  $p_2 = \$12$  and is producing the quantity  $q_2 = 80$ . Assume that Firm 1 takes Firm 2's price of \$12 to be unaffected by its own decision; determine Firm 1's residual demand curve, marginal revenue curve, and profit-maximizing output level and price. Now assume instead that Firm 1 takes Firm 2's output of 80 units to be unaffected by its own decision; determine Firm 1's residual demand curve, marginal revenue curve, and profit-maximizing output level and price. You should find that everything is different in the first case than in the second case.

## Game Theory

7.1 In a Cournot duopoly each firm chooses its profit-maximizing quantity, taking as given the quantity produced by its rival. Assume that each firm's reaction function  $r_i(\cdot)$ , is a well-defined single-valued function on  $\mathbb{R}_+$ ; assume that each  $r_i$  is continuous and nonincreasing; and let  $\beta_i$  denote  $r_i(0)$  — i.e.,  $\beta_i$  is the output firm  $i$  chooses if its rival produces zero.

- (a) Prove that a Cournot (*i.e.*, Nash) equilibrium exists.
- (b) Give an example to show that the assumptions above are not enough to ensure that the Nash equilibrium is unique. (Give a simple diagrammatic example, not a worked out analytical example.)
- (c) Use your example in (b) to explain why, if there are multiple equilibria, the comparative statics (*i.e.*, the effects of a shift in one or both reaction functions) cannot be the same at all equilibria.

7.2 Suppose the Prisoners' Dilemma game

	D	C
D	3,3	5,1
C	1,5	4,4

will be played twice in succession by the same pair of players, and that each player will observe the other's first-stage play before the second stage is played. Each player's payoff in the repeated game is the sum of the payoffs he receives at the two stages of play.

(a) Draw the game tree for the repeated game, *i.e.*, the game that consists of two successive plays of the PD game given above.

(b) In the normal form (or strategic form) of this RPD, indicate why each player has  $2^5 = 32$  available strategies. (You needn't actually write down each of the strategies.)

Since there are 32 strategies for each player, the payoff matrix for the normal form description of this RPD has  $32 \times 32 = 1024$  cells, or strategy profiles. But we can simplify things. The fact that a strategy tells a player how his play at stage 2 should depend on *his own* play at stage 1 is redundant: a strategy needs to specify only what to choose at stage 1 and then what to do at stage 2 as a function of what *the other player* did at stage 1. (To put it another way, if my strategy tells me to choose C at stage 1, for example, then there's no need for it to also tell me what to do at stage 2 as a function of what I did at stage 1.)

(c) Make use of the observation above to enumerate the eight behaviorally distinct strategies for each player. Write down the  $8 \times 8$  normal form payoff bi-matrix for the game, placing the two players' payoffs in each of the 64 cells.

(d) Can you identify any of a player's eight strategies as the Tit-for-Tat strategy? Can you give a similar informal descriptive name to any of the other strategies?

(e) If a player is using the Tit-for-Tat strategy, what is the best strategy for the other player to use in response?

(f) Determine each player's best-response correspondence in the payoff bi-matrix.

(g) Use the best-response correspondences in (f) to determine whether either player has a dominant strategy.

(h) Determine all equilibrium profiles of strategies. (You should be able to do this by finding all intersections of the two best-response correspondences in (f).) What are the players' payoffs in each of the equilibria? How will the path of play proceed in the various equilibria — *i.e.*, in each equilibrium, what profile of actions will we observe the players choosing at each stage of play? Are any of the equilibrium strategies weakly dominated?

(i) Suppose Player 1 believes that with probability  $p$  Player 2 will play the Tit-for-Tat strategy, and that with probability  $1 - p$  Player 2 will play the strategy “always defect” (*i.e.*, will always choose D). What values of  $p$  would induce Player 1 to play the “reputation strategy” in which she plays C at the first play and then D at the second play?

(j) Now suppose the players are going to play this same PD game three times. Indicate how you would go about extending the two-play game tree to yield the three-play tree. (Don't actually draw the tree. Just explain how you would extend the two-play tree to get it.) Suppose again that Player 1 believes that with probability  $p$  Player 2 will play the Tit-for-Tat strategy, and that with probability  $1 - p$  Player 2 will play “always defect.” Player 1 is trying to decide whether to always defect or to instead play Tit-for-Tat until the last play (at which Player 1 will defect). Thus, Player 1 has to consider four possible profiles of strategies. For each of the four profiles, determine the resulting path of play and the resulting payoff for Player 1. Determine how large  $p$  would have to be to induce Player 1 to play Tit-for-Tat until defecting on the last play instead of simply always defecting.

(k) Now suppose the players are going to play this same PD game  $T$  times. (If you can't see how to answer this question for general values of  $T$ , try it for  $T = 4$ . You've already done it for  $T = 2$  and for  $T = 3$ , above.) For each of the four strategy profiles in (j), determine the resulting path of play and the resulting payoff for Player 1. Determine how large  $p$  would have to be to induce Player 1 to play Tit-for-Tat until defecting on the last play instead of simply always defecting. (You might find it easier if you take as a benchmark Player 1's total payoff when both players always defect, and then figure out how much larger or smaller Player 1's total payoff will be under each of the other three strategy profiles.)

(l) As the repeated play of the PD game proceeds, will Player 1 be able to “update her belief”  $p$  about which strategy Player 2 might be using, on the basis of her observation of Player 2's actions?

## Public Goods and Other Externalities

8.1 Ms. Alpha and Mr. Beta live together. Each cares only about the cleanliness of the house they share and about the simoleans he or she consumes. Denote the level of cleanliness by  $x$ , and denote Ms. Alpha's and Mr. Beta's consumption of simoleans by  $y_A$  and  $y_B$ . Ms. Alpha's utility function is  $u(x, y) = \min\{2x, y_A\}$  and Mr. Beta's is  $u(x, y) = \min\{x, y_B\}$ . It's only possible to convert simoleans into cleanliness at a rate of one simolean for each unit of cleanliness. If no simoleans are devoted to cleaning the house, the resulting level of cleanliness is  $x = 0$ . Ms. Alpha and Mr. Beta are endowed with a total of 120 simoleans; therefore the feasible allocations are the ones that satisfy  $x + y_A + y_B = 120$ .

(a) Four alternative allocations are described below. For each of the allocations do the following: Determine if the allocation is Pareto optimal; if a Pareto improvement exists, find a Pareto optimal allocation that makes *both* people strictly better off.

$$(a1) \quad (x, y_A, y_B) = (30, 60, 30)$$

$$(a2) \quad (x, y_A, y_B) = (60, 20, 40)$$

$$(a3) \quad (x, y_A, y_B) = (40, 50, 40)$$

$$(a4) \quad (x, y_A, y_B) = (36, 40, 36)$$

(b) Draw the utility-possibility frontier for these Ms. Alpha and Mr. Beta.

Assume for (c) and (d) that ownership of the 120-simolean endowment is divided equally between Ms. Alpha and Mr. Beta.

(c) Can the allocation (a1) above be supported as a “voluntary-contributions” equilibrium – *i.e.*, is the allocation a noncooperative equilibrium if the household's outcome is determined by each person voluntarily contributing simoleans to be used for cleaning? Determine the set of all voluntary-contributions equilibria.

(d) Can the allocation (a1) above be supported as a Lindahl equilibrium? Determine the set of all Lindahl equilibria.

8.2 There are two consumers, Ms. A and Ms. B, and two goods, quantities of which will be denoted by  $x$  and  $y$ . The  $x$ -good is a pure public good which can be produced at a constant marginal cost of  $c$  units of the  $y$ -good for each unit of the  $x$ -good produced. Each consumer's preference for the two goods has the same form, differing only in a parameter  $\alpha$ : the consumer's marginal rate of substitution between the goods is 3 when  $x < \alpha$  and is 1 when  $x > \alpha$ . The following utility function can be used to describe these preferences:

$$u(x, y) = \begin{cases} y - \alpha - x & = x + y - \alpha, & \text{if } x \geq \alpha \\ y + 3(x - \alpha) & = 3x + y - 3\alpha, & \text{if } x \leq \alpha. \end{cases}$$

The parameter values for Ms. A and Ms. B satisfy  $0 < \alpha_A < \alpha_B < 10$ . There are 100 units of the  $y$ -good to be used for consumption and/or production of the public good. An interior allocation is defined to be one in which each consumer receives a positive amount of the  $y$ -good.

- (a) Let  $c = 5$ . Determine all the Pareto optimal allocations, if there are any. For every interior allocation  $(x, y_A, y_B)$  that is not Pareto optimal, find an allocation that is both Pareto optimal and a Pareto improvement upon  $(x, y_A, y_B)$ .
- (b) Again let  $c = 5$ . Determine all non-interior Pareto optimal allocations, if there are any. Explain why your answer is the right one.
- (c) Let  $c = 2$ . Determine all the interior Pareto optimal allocations.
- (d) Measuring  $c$  on the horizontal axis and  $x$  on the vertical axis, draw a diagram indicating, for each level  $c$  of marginal cost, the associated public good levels that are consistent with Pareto optimality at interior allocations.
- (e) Let  $c = 2$ . Suppose that production of the public good results only from Ms. A and Ms. B voluntarily contributing some of their private-good holdings as input to the public good production process. Let  $z_A$  and  $z_B$  denote the contributions of private good by Ms. A and by Ms. B. Draw the two women's reaction curves and determine the Nash equilibrium. Is the Nash equilibrium allocation Pareto optimal?

8.3 A certain restaurant in town is known for refusing to give separate checks to customers. After a group has ordered and eaten together at this restaurant, the group is presented with a single check for the entire amount that the group has eaten. It has been suggested that the restaurant does this because, with a single check, those who dine in groups will be more likely to simply divide the charge equally, each person paying the same amount irrespective of who ordered the most; and that diners, knowing they will ultimately divide the charge equally, will order more than they would have ordered had each expected to pay only for his own order. Analyze this situation using the following model.

There are  $n$  diners in a group. Each has a utility function of the form  $u_i(x_i, y_i) = y_i + a_i \log x_i$ , where  $x_i$  represents the amount of food (in pounds) ordered and eaten by  $i$ , and  $y_i$  represents the amount of money that  $i$  has after leaving the restaurant. The restaurant charges  $p$  dollars for each pound of food, and the restaurant's profit is an increasing function of the amount of food that it sells at the price  $p$ . Each diner knows when he orders his food that the group will divide the check equally when it is time to pay.

Compare the outcome under this check-splitting arrangement with the outcome when each diner pays for his own order. Compare, in particular, the restaurant's profit in each case and the diners' welfare in each case. Is there an alternative arrangement that will make the diners better off than in either of these arrangements?

8.4 A group of  $n$  students has determined that they're in deep trouble in their economics course, so they have decided to hire a tutor to give them a review session. The tutor will charge them  $p$  dollars per hour. Each member of the group has a utility function of the form  $u_i(x, t_i) = -t_i + a_i \log x$ , where  $x$  denotes the length of the review session, in hours, and where  $t_i$  denotes the amount he has to pay to the tutor. Assume that  $a_1 > a_2 > \dots > a_n$ . The group has agreed to decide on the length of the session by the following method: Each member of the group will announce his vote, a non-negative real number  $m_i$ ; the length of the session will be the average (i.e., the mean) of all  $n$  votes; and each member will pay the same amount to the tutor — i.e., they'll share the cost of the tutor equally.

(a) What will be the outcome of this decision procedure, assuming that each member of the group knows the others' preferences and how the others will vote?

(b) Determine all the Pareto optimal allocations

8.5 Ms. Alpha and Mr. Beta have just terminated their marriage. They have agreed that Mr. Beta will raise their only child, little Joey Alpha-Beta. The two parents hold no animosity toward one another, and each is intensely concerned about little Joey’s welfare. Their preferences are described by the utility functions

$$u^A(x, y_A) = x^\alpha y_A \quad \text{and} \quad u^B(x, y) = x^\beta y_B,$$

where  $y_A$  and  $y_B$  denote the number of dollars “consumed” directly by the respective parents in a year, and  $x$  denotes the number of dollars per year consumed by Joey. Joey’s consumption is simply the sum of the support contributions from his mother and father,  $s_A + s_B$ . These contributions will be voluntary: neither parent has sought a legal judgment against the other. Assume throughout that  $\alpha = 1/4$  and  $\beta = 1/3$ .

(a) Suppose Joey’s mother is unable to contribute anything toward Joey’s support, so that Mr. Beta must provide, out of his \$40,000 annual income, for both his own consumption,  $y_B$ , and Joey’s consumption,  $x$ . Express Mr. Beta’s budget constraint both analytically, and diagrammatically. Determine Mr. Beta’s marginal rate of substitution between  $x$  and  $y_B$  at the choice he will make, and draw a diagram representing his choice problem. What levels of  $x$  and  $y_B$  will Mr. Beta choose?

(b) Actually, Ms. Alpha is going to contribute to Joey’s support, but she is going to observe how much Mr. Beta contributes,  $s_B$ , and then choose her contribution,  $s_A$ . Suppose Mr. Beta does the same — i.e., each parent takes the other’s contribution as given. If Ms. Alpha’s annual income is \$48,000 and Mr. Beta’s is \$40,000, what will be their equilibrium contributions to Joey’s support?

(c) Find an allocation of the parents’ incomes that will make them both happier than the one in (b).

(d) Determine the two equations (viz., the marginal condition and the “on-the-constraint” condition) that characterize the Pareto optimal allocations.

(e) Indicate some of the difficulties that a neutral third party (e.g., a judge) might encounter in attempting to implement some method for arriving at a Pareto optimal allocation of the parents’ incomes.

8.6 Three farmers (labeled  $i = 1, 2, 3$ ) have recognized that any fertilizer sprayed in their neighborhood is a public good to them. Fertilizer costs \$3 per gallon. The farmers' profits, as functions of the amount  $x$  of fertilizer sprayed, are given by the functions  $\pi_i(x) = \alpha \ln x$ , where  $\alpha_1 = 1$ ,  $\alpha_2 = 2$ , and  $\alpha_3 = 3$ . (The  $\pi_i$  functions give the farmers' respective profits, in dollars, *not counting what they pay for fertilizer*.) Each farmer is interested only in maximizing the profit he will be left with after deducting his payment for fertilizer. An allocation here is a list  $(x, t_1, t_2, t_3)$  specifying how much will be sprayed (that's  $x$ ) and how much each farmer will pay (that's  $t_i$  for farmer  $i$ ). Efficiency clearly requires that  $t_1 + t_2 + t_3 = 3x$ .

(a) Determine which interior allocations are Pareto optimal.

(c) The farmers have agreed to use the following method to determine this month's allocation of fertilizer and payments: each farmer will place a *request*  $r_i$  with the spraying company; the company is then authorized to spray  $x = r_1 + r_2 + r_3$  gallons and to charge the farmers the amounts

$$t_1 = (1 + r_2 - r_3)x \quad t_2 = (1 + r_3 - r_1)x \quad t_3 = (1 + r_1 - r_2)x.$$

Notice that  $t_1 + t_2 + t_3 \equiv 3x$ . Determine the Nash equilibrium of this scheme (i.e., assume that each farmer chooses his request taking the other two requests as given).

8.7 Three housemates, Amy, Bev, and Cathy are about to buy a satellite dish. They must decide how large a dish to buy. Their preferences are as follows, where  $x$  denotes the diameter of the dish (in meters) and  $t_i$  denotes the amount that person  $i$  pays (in dollars):

$$u_i(x, t_i) = \alpha_i \log x - t_i, \quad i = A, B, C, \quad \text{and} \quad \alpha_A < \alpha_B < \alpha_C.$$

Satellite dishes cost  $\beta$  dollars per meter of diameter (i.e., a dish of diameter  $x$  meters costs  $\beta x$  dollars).

(a) Which decisions  $(x, t_A, t_B, t_C)$  are Pareto efficient?

(b) The housemates have decided to use the following procedure to decide upon  $x$  and  $t_A$ ,  $t_B$ , and  $t_C$ : each person will cast a vote for the size dish she would like; they will buy a dish the size of the median of the three votes; and they will divide the cost of the dish equally. Votes must be non-negative real numbers. Determine the Nash equilibrium (or, if there are multiple Nash equilibria, determine them all), and indicate how the equilibrium outcome(s) compare to the efficient outcomes.

(c) Does anyone have a dominant strategy? Explain.

8.8 A community with  $n$  households is contemplating improving its roads. Let  $x$  denote the level of improvement; any non-negative  $x$  can be chosen, but the cost of improvement level  $x$  will be  $cx$  dollars, where  $c$  is a positive number. Households do not all have the same preferences; denote household  $i$ 's preference by the utility function  $u_i(x, y_i)$ , where  $y_i$  denotes the amount of money (in dollars) the household has available to spend after the road improvements have been paid for (thus, no  $y_i$  can be negative). Derive the Samuelson marginal condition of Pareto efficiency . . .

- (a) for allocations in which all  $y_i$  are positive, and . . .
- (b) for allocations in which one or more  $y_i$  is zero.

8.9 The tiny country of DeSoto has  $n$  households, each of which owns a car. The residents of DeSoto have only two interests in life — driving their cars and consuming the economy's only tangible commodity, simoleans. Each household has a utility function of the form

$$u_i(x_i, y_i) = y_i + v_i(x_i) - a_i H,$$

where  $y_i$  denotes consumption of simoleans,  $x_i$  denotes miles driven, and  $H$  denotes the level of hydrocarbons in the air. Cars use simoleans for fuel: every mile that a car is driven uses up  $c$  units of simoleans, but the burning of simoleans also puts  $b$  units of hydrocarbons into the air for every mile that the car is driven. In other words,  $H = (x_1 + \dots + x_n)b$ . Use  $A$  to denote the sum of all households'  $a_i$  parameters and  $X$  to denote the total of all miles driven by all households, and assume that each function  $v_i$  is strictly concave and increasing. Consider only those allocations in which each  $x_i$  and each  $y_i$  is positive. Each household has a positive endowment of simoleans.

- (a) Give the  $n$  marginal conditions that characterize the Pareto optimal allocations, and interpret them in words.
- (b) Give the  $n$  marginal conditions that characterize the Walrasian equilibrium, and interpret them in words.
- (c) Determine whether, in the equilibrium, all families necessarily drive “too much,” all families necessarily drive “too little,” or whether the miles driven might be either too large or too small depending upon the data of the problem.

8.10 The de Beers Brewery uses water from the Pristine River in its brewing operations. Recently, the United Chemical Company (also called Chemco) has opened a factory upstream from de Beers. Chemco's manufacturing operations pollute the river water: let  $x$  denote the number of gallons of pollutant that Chemco dumps into the river each day. De Beers's profits are reduced by  $x^2$  dollars per day, because that's how much it costs de Beers to clean the pollutants from the water it uses. Chemco's profit-maximizing level of operation involves daily dumping of 30 gallons of pollutant into the river. Altering its operations to dump less pollutant reduces Chemco's profit: specifically, Chemco's daily profit is reduced by the amount  $(1/2)(30 - x^2)$  if it dumps  $x$  gallons of pollutant per day. There are no laws restricting the amount that Chemco may pollute the water, and no laws requiring that Chemco compensate de Beers for the costs imposed by Chemco's pollution.

(a) Coase's argument holds that the two firms will reach a bargain in which the Pareto efficient level of pollution will be dumped. Determine the efficient level of pollution. If efficiency requires that  $x < 30$ , then determine the range of the bargains the two firms could be expected to reach — i.e., the maximum and minimum dollar amounts that de Beers could be expected to pay to Chemco in return for Chemco's agreeing to dump only  $x$  (less than 30) gallons per day.

(b) Now suppose a law is passed that requires anyone who pollutes the Pristine River to fully compensate any downstream firm for the damages caused by the polluter's actions. How does this change the Pareto efficient level of pollution? How does this change the pollution level that Chemco and de Beers will agree to? How does it change the payments that one of the firms will make to the other?

(c) Now suppose that de Beers is not the only firm harmed by Chemco's pollution: there are more than one hundred firms whose profits are reduced by the pollution. How would this affect your answers in (a) and (b)? (You will not be able to give a precise quantitative answer here, because you do not know exactly how much each firm is damaged by the pollution. But describe qualitatively how the answers to (a) and (b) will change.)

8.11 The Simpsons and the Flanders are next-door neighbors. The Simpsons enjoy listening to music, played very loud. The Flanders prefer quiet. Using  $x$  to denote the volume of the Simpsons' music in "decibooms" (tens of decibels), and using  $y_S$  and  $y_F$  to denote their monthly consumption of other goods (in dollars' worth), the preferences of the Simpson and Flanders households are described by the following utility functions:

$$u_S(x, y_S) = y_S + 9x - \frac{1}{2}x^2 \quad \text{and} \quad u_F(x, y_F) = y_F - x^2.$$

Each family's monthly income is \$3,000. Note that their marginal rates of substitution are given by  $MRS_S = 9 - x$  and  $MRS_F = -2x$ .

(a) Determine the Pareto efficient volume of the Simpsons' music.

Suppose there is no law restricting the volume at which people can play music; thus, the Simpsons have the right to play their music at whatever volume they like. According Coase's argument, the two families will reach an agreement about the volume of the music, and one family will pay some monetary compensation to the other.

(b) According Coase's argument, what volume of music will the families agree upon? Which family will receive compensation?

(c) Determine the Lindahl equilibrium – the music volume and the amount of money that one family transfers to the other, again assuming that there is no law restricting the volume of music.

(d) Determine the core outcome when there is no law restricting music volume.

(e) Now suppose a law has been passed that imposes a \$500 fine on anyone whose neighbor justifiably complains about loud music. According to Coase's argument, what volume of music will the Simpsons and Flanders now agree to? What is the Lindahl outcome? Determine a core outcome.

(f) Now suppose there are 20 students living in a dormitory, half of whom have preferences described by  $u_S$  above, and half of whom have the preferences described by  $u_F$ . Everyone has a stereo, and  $x$  is the volume of the stereo that is played the loudest. There is no restriction on the volume at which stereos may be played. What is the Pareto efficient level of  $x$ ? What is the Lindahl outcome? Is Coase's solution more likely than with only two individuals, or less likely? Why?

8.12 Ann, Bob, and Carol are renting a house together and they must decide what level of cable TV service they will subscribe to. They can choose any non-negative level of service  $x$ , for which they will be charged  $6x$  dollars. Unfortunately, the three roommates do not have the same preferences for cable TV service: each one's preferences are described by a utility function of the form  $u_i(x, y_i) = y_i + r_i \log x$ , where  $y_i$  denotes dollars available to spend on other goods, and the values of  $r_i$  for Ann, Bob, and Carol are  $r_A = 4$ ,  $r_B = 8$ , and  $r_C = 36$ . Each of the roommates is endowed with 40 dollars.

(a) Determine all service levels that are consistent with Pareto optimality when every  $y_i > 0$  ( $i = A, B, C$ ).

(b) Determine the Lindahl allocations and prices.

(c) Suppose the housemates use the following procedure to determine the service level  $x$  they will purchase and how they will pay for their purchase: each housemate will announce (or "vote for") the service level  $m_i$  he or she claims to most prefer; then they will purchase a service level  $x$  equal to the median of the three votes, and they will share the cost,  $6x$ , equally (i.e., each will pay  $2x$ ).

(c') Suppose  $m_A = 2$  and  $m_B = 4$ . Draw Carol's "budget set" – i.e., the set of all bundles  $(x, y_C)$  Carol can obtain for herself via her vote  $m_C$ , assuming that Ann and Bob don't change their votes. Which bundle will she choose, and what vote(s) could she cast in order to achieve that bundle?

(c'') Suppose  $m_A = 2$  and  $m_C = 12$ . Draw Bob's budget set. Which bundle will he choose, and what vote(s) could he cast in order to achieve that bundle?

(c''') Determine the Nash equilibrium allocation(s).

8.13 A large university assigns three graduate students to each office. Each office has a thermostat by which the temperature in the office can be set at any level from  $60^\circ$  to  $90^\circ$  Fahrenheit. The university recognizes that the three office-mates generally will not prefer the same temperature, and to avoid arguments and lawsuits the university is going to mandate a rule by which office-mates are to decide on the temperature for their office. Two rules have been proposed, each of which requires each office-mate to state which temperature he desires. These “votes” are required to be not less than 60 nor greater than 90. Denote student  $i$ 's vote by  $m_i$ . The rules differ in the way the three votes are used to determine the temperature:

The Median Rule: The temperature will be set at the median of the three votes.

The Mean Rule: The temperature will be set at the mean of the three votes.

Assume that each graduate student's preference for alternative temperatures can be described by a strictly concave real function  $u_i$  on the interval  $[60,90]$ : student  $i$  prefers temperature  $x$  to temperature  $y$  if  $u_i(x) > u_i(y)$ . Let  $\beta_i$  denote the temperature student  $i$  likes best (i.e., the maximizer of  $u_i$ ). Note that  $\beta_i \in [60, 90]$ . Without loss of generality, assume that  $\beta_1 \leq \beta_2 \leq \beta_3$ .

(a) Verify that each student has a dominant strategy if the Median Rule is used. In any given office, will these dominant strategies be the unique Nash equilibrium?

(b) Suppose the Mean Rule is used and consider an office in which  $\beta_2$ , the median value of  $\beta_i$ , is at least 80. (Note that this is the median most-preferred temperature, not necessarily the median vote.) Also assume that  $\beta_1 < \beta_2 < \beta_3$ . Determine a Nash equilibrium in this office. Is it the unique Nash equilibrium?

(c) Determine which, if either, of the outcomes in (a) and (b) are Pareto optimal.

(d) Now assume that each student's preference is described by a utility function of the form  $U_i(x, y_i) = y_i + u_i(x)$ , where  $u_i$  is as described above, and where  $y_i$  denotes the amount of money  $i$  has. Assume, moreover, that it's possible to transfer money from one student to another. How will this change your answers to the questions posed in (a), (b), and (c)?

8.14 100 men have access to a common grazing area. Each man can choose to own either no cows, one cow, or two cows to provide milk for his family. The more cows the grazing land is required to support, the lower is each cow's yield of milk. Specifically, each man obtains

$$Q_i = (250 - X)x_i \text{ quarts of milk per year,}$$

where  $x_i$  denotes the number of cows the man owns, and  $X$  denotes the total number of cows owned by all 100 men (i.e.,  $X = x_1 + \dots + x_{100}$ ). Each man wants to obtain as much milk as he can, but no man has the resources to own more than two cows.

(a) How many cows do you predict each man will own? Explain your prediction. Indicate, in particular, whether your prediction is some sort of equilibrium, and if so whether it is a unique equilibrium, and whether this equilibrium is one that would be likely to be reached quickly, or only after a long period of time during which the men learn how one another behaves. If your prediction is not some sort of equilibrium, explain why you have predicted as you have.

(b) Assume that men can make transfers of milk among themselves (in particular, that men with more cows can give milk to those with fewer cows to compensate them for owning fewer cows). Is your prediction in (a) Pareto efficient for the 100 men? If so, verify it. If not, then find a Pareto optimal allocation of milk to the men that makes everyone strictly better off, and a pattern of cow ownership and transfer payments (in quarts of milk) that will support that allocation.

(c) Now suppose that there are only two men whose cows share a common grazing area, and that again each man can choose to own either no cows, one cow, or two cows. Each cow's daily yield of milk, in quarts, depends on how many cows in total are grazing, as follows:

Total cows grazing:	1	2	3	4
Each cow's daily yield:	8	5	3	2

What are the Pareto efficient individually rational allocations of milk (recall that an allocation is "individually rational" if each man is at least as well off as he would be by "unilateral" action)? What are all the patterns of cow ownership and transfer payments that support these allocations? Determine all the core allocations of milk to the two men.

(d) For the situation described in part (c), answer all the questions posed in (a).

8.15 Ozone City, located on the Left Coast, has  $n$  residents, all of whom do a lot of driving. A simple model of the situation has only two goods, gasoline (gallons denoted by  $x$ ) and dollars (quantities denoted by  $y$ ). The market for gasoline is competitive, and it costs the typical firm  $\beta$  dollars to deliver a gallon of gasoline at the pump. All the gasoline combustion produced by all the driving causes serious pollution of Ozone City's air: the pollution level, denoted by  $s$ , is given by the equation  $s = \alpha x$ , where  $x$  is the total gallons of gasoline sold (all of which is used in driving). Each resident  $i$ 's preferences are described by a differentiable utility function  $u^i(x_i, y_i, s)$ , where  $x_i$  denotes the gallons of gasoline he buys and  $y_i$  the number of dollars he consumes. Of course, the partial derivatives of  $u^i$  satisfy  $u_x^i > 0$ ,  $u_y^i > 0$ , and  $u_s^i < 0$ .

(a) Derive the marginal conditions that characterize the Pareto efficient outcomes.

For the remainder of this problem, assume that each resident, when making his decision, ignores the effect of his own purchase,  $x_i$ , upon the total  $x = \sum x_j$ .

(b) Determine the marginal conditions that the market outcome will satisfy if there are no interventions such as taxes or subsidies. Can you determine, without knowing the specific utility functions, whether the market outcome involves "too much" or "too little" driving?

(c) Determine a tax-and-rebate arrangement that would induce a Pareto efficient outcome via individual market decisions. Describe any difficulties one would likely encounter in implementing the tax-and-rebate arrangement.

(d) This model will not allow one to analyze the individual's decision whether to purchase a more fuel-efficient car. How would you change the model to allow this kind of analysis?

8.16 There are  $n$  people in the economy, and only two goods that they care about consuming, food and leisure. Each person owns a machine that produces  $Kz$  units of food if someone gives up  $z$  of his leisure hours to operate the machine; the coefficient  $K$  is the amount of knowledge in the economy. There is a third use to which a person can put his time (in addition to working and consuming his time in the form of leisure): he can spend his time “adding to knowledge.” Everyone’s production coefficient,  $K$ , is equal to the sum of the knowledge gained by all the members of the economy. (To keep things simple, assume that the economy only operates once; equivalently, in each market period all knowledge from previous periods is forgotten.)

Use the following notation:  $x_i$  is  $i$ ’s consumption of food;  $y_i$  is  $i$ ’s consumption of leisure (hours);  $z_i$  is the number of hours  $i$  works producing food, either with his own machine or with others’ machines;  $k_i$  is the number of hours  $i$  devotes to gaining knowledge;  $K = \alpha \sum_1^n k_i$ ; and  $p$  is the market price of food. The market price of labor is \$1.

Determine each of the following:

- (a) The constraints that characterize the feasible allocations.
- (b) The first-order conditions that characterize the Pareto optimal allocations.
- (c) The constraint(s) imposed upon an individual by competitive markets, assuming that there is no market for knowledge – any knowledge that an individual gains is automatically in the public domain.
- (d) The first-order conditions that characterize the individual’s choice in the marketplace.
- (e) Derive whatever economic implications you can from the first-order conditions in (b) and (d) – give as complete an analysis of the situation as you can.
- (f) How would the market outcome be changed if there is only one machine and it’s in the public domain – i.e., a single machine into which anyone can put  $z$  hours of work and obtain  $Kz$  units of food in return? What if the single machine is owned by just one of the individuals in the economy?

8.17 Activities that generate negative externalities, such as pollution, will generally be carried out at a level greater than Pareto efficiency would require. On the other hand, it is often argued, reducing the externality-generating activity will result in a loss of jobs. You should now be able to produce some insight into this issue by constructing your own simple model. Combine the ideas in your simple model of a pollution-generating activity with the ideas in the exercises you have done that concern the welfare differences between monopoly and competition. As in the pollution model, let  $s$  denote the level of pollution, let  $x_i$  denote the amount  $i$  consumes of the pollution-generating product, and let  $y_i$  denote the amount  $i$  consumes of the good that is also used as input in the production of the  $x$ -good – and, in particular, let this latter good be  $i$ 's “leisure (non-working) time,” so that the amount  $z_i = \overset{\circ}{y}_i - y_i$  is the amount of time he sells as an employee to the producers of the  $x$ -good. Make up a numerical example (I suggest using a constant-returns-to-scale production technology) in which you can determine the outcome and utility levels determined by Pigovian taxes and transfers (ignoring the incentive issues associated with determining the taxes and transfers). Determine, in particular whether, by “gainers” compensating “losers,” a reduction in the pollution-generating activity can be a Pareto improvement. (A more complete model would include two produced products (one polluting, one not) and two kinds of labor used in the production process (some consumers endowed with one kind of labor and other consumers with the other kind). If we moved from the unregulated market to the Pigovian outcome, what would happen to the output levels of the two products, to the incomes of the two types of labor, and to their utility levels?

8.18 Acme Nurseries and Badweiser Brewery are located adjacent to one another. Each imposes an external cost on the other: the fertilizer that Acme uses increases Badweiser's costs, and the air pollution from Badweiser's production increases Acme's costs. Specifically, if  $x_A$  and  $x_B$  are Acme's and Badweiser's production levels, then their profits (in dollars per hour) are given by the functions  $\pi_A$  (for Acme) and  $\pi_B$  (for Badwesier):

$$\pi_A(x_A, x_B) = (30 - x_B)x_A - x_A^2 \quad \text{and} \quad \pi_B(x_A, x_B) = (30 - x_A)x_B - x_B^2.$$

- (a) On a single diagram draw the two firms' reaction functions. Calculate the Nash equilibrium, assuming that each firm takes the other's production level as given.
- (b) If the firms were somehow able to choose their production levels cooperatively (for example, if they were owned by the same person), what would those levels be?
- (c) Suppose the firms are not owned by the same person. Describe the Coase "Theory of Social Cost" argument as it applies to this situation, and determine the outcome(s) predicted by the Coase argument – the production levels and any payments from one firm to the other.
- (d) Consider two possible situations: In one case the two firms make their production decisions once and for all, at a single date; in the other case, the two firms make their production decisions repeatedly, day after day, year after year. Would the Coase argument be more likely in one situation than the other, and if so, why? Would your answer be the same if this were a so-called "pecuniary" externality – for example, if the firms were Cournot duopolists, selling an identical, costless-to-produce product in a market where the price is  $p = 30 - (x_A + x_B)$ , so that the externality occurs through the effects on revenue instead of on costs?
- (e) Suppose there is a law stating that any firm polluting the water must fully reimburse any firm whose costs are increased by that pollution, but that there is no such law covering air pollution. Determine the outcome(s) predicted by the Coase argument – the production levels and any payments from one firm to another. What if the law states that the polluting firm need only reimburse half of the costs it imposes on others?

## Time, Uncertainty, and Incomplete Markets

9.1 Suppose half the people in the economy choose according to the utility function

$$u^A(x_0, x_H, x_L) = x_0 + 5x_H - .3x_H^2 + 5x_L - .2x_L^2$$

and the other half according to the utility function

$$u^B(x_0, x_H, x_L) = x_0 + 5x_H - .1x_H^2 + 5x_L - .2x_L^2$$

where

$x_0$  represents consumption “today,”

$x_H$  represents consumption “tomorrow” in event  $H$ , and

$x_L$  represents consumption “tomorrow” in event  $L$ .

Storage of the consumption good from today until tomorrow is not possible. Each person is endowed with twelve units of the good in each of the two periods, no matter which of the two possible events occurs.

In your answers, consider only allocations that give all type  $A$  people the same consumptions and all type  $B$  people the same consumptions, so that you will be able to completely describe an allocation with the six variables  $x_0^A, x_H^A, x_L^A, x_0^B, x_H^B, x_L^B$ .

- (a) Which allocations are Pareto optimal?
- (b) Determine the Arrow-Debreu equilibrium — *i.e.*, the Arrow-Debreu prices and allocation.
- (c) Suppose that the only market is a credit market (*i.e.*, a market for borrowing and lending). There are no markets in which one can insure oneself against either of tomorrow’s two possible events. What will be the competitive equilibrium interest rate and how much will each person borrow or save? Is the equilibrium allocation Pareto optimal?
- (d) In addition to the credit market in (c), suppose there is another market as well, in which one can buy or sell insurance today against the occurrence of event  $H$ . Each unit of insurance that a person purchases is a contract in which the seller of the contract agrees to pay the buyer one unit of consumption tomorrow if event  $H$  occurs. Let  $p$  denote the market price of the insurance: the buyer pays the seller  $p$  units of consumption today for each unit of insurance he purchases. Determine the competitive equilibrium prices (*i.e.*, the interest rate and the price  $p$  of insurance) and the equilibrium allocation.

9.2 Alice and Bill each have fifteen dollars today, and each will also have fifteen dollars tomorrow. Before tomorrow arrives an election is going to take place. Bill knows that if the Democrats win the election there will be lots of parties with lots of celebrities; because he's such a party animal, Bill would like to have more money in the event that the Democrats win, in order to enable him to attend all the parties. Alice is an economist and does not attend parties, so her intertemporal preferences do not place as much weight on the event that the Democrats win. Specifically, Alice's and Bill's intertemporal utility functions are

$$u_A(x_{A0}, x_{AD}, x_{AR}) = x_{A0} + 9x_{AD} - .4x_{AD}^2 + 12x_{AR} - .4x_{AR}^2$$

$$u_B(x_{B0}, x_{BD}, x_{BR}) = x_{B0} + 9x_{BD} - .2x_{BD}^2 + 12x_{BR} - .4x_{BR}^2 ,$$

where  $x_{i0}$  denotes dollars consumed today by  $i$ , and  $x_{i\theta}$  denotes dollars consumed tomorrow by  $i$  in state  $\theta$ .

(a) Determine the Arrow-Debreu prices and allocation(s) for the economy consisting of just Alice and Bill. What is the interest rate?

(b) Suppose the only markets open today in which one can contract for dollars tomorrow are two security markets. Security Gamma returns one dollar tomorrow in each state; Security Delta returns one dollar tomorrow if the Democrats win, but requires the holder to pay a dollar tomorrow if the Republicans win. (Gamma securities are generally sold by banks; Delta securities are generally sold by bookmakers.) What are the equilibrium prices (today) of these two securities? How many of each will Alice and Bill buy?

9.3 You're teaching an undergraduate intermediate economics course and you must design a lecture on uncertainty and insurance markets. Your class has already learned about the concept of general equilibrium of markets and about Pareto efficiency, and you've used the Edgeworth box device to help teach these ideas. Describe the simplest possible model you could use (two people, one good, two possible states of the world, no consumption before the uncertainty is resolved) to demonstrate that the market outcomes will generally be inefficient if there are uncertainties for which no markets exist for individuals to "trade risk" with one another.

9.4 Either the Republicans or the Democrats will win the next election — *i.e.*, one of the two states of the world  $\theta = R$  or  $\theta = D$  will occur. Apu and Bart are each endowed with ten pesos today; each will also be endowed, for certain, with fifteen pesos tomorrow. Each one's preferences are described by a von Neumann-Morgenstern utility function of the form

$$u(c_0, c_R, c_D) = c_0 + E(5 \log c_\theta) = c_0 + 5\pi_R \log c_R + 5\pi_D \log c_D,$$

where  $c_0$  denotes pesos consumed today;  $c_\theta$  denotes consumption of pesos in state  $\theta$ ; and  $\pi_\theta$  denotes the individual's subjective probability assessment that state  $\theta$  will occur. Apu believes that the two states are equally likely to occur, but Bart believes there is a 3/4 chance that the Republicans will win.

(a) Determine the interior allocations that are Pareto optimal.

For parts (b), (c), and (d) assume that Apu and Bart are the only two traders and that each one behaves as a price-taker in all markets.

(b) Assume that the only market available is a borrowing and lending market. What will the equilibrium interest rate be, and how much will each person save or borrow? How would your answers change if the individuals' subjective probabilities were different?

(c) Assume that there are complete Arrow-Debreu contingent claims markets. Determine the equilibrium prices and consumption levels. What is the implicit interest rate?

(d) Now suppose that the only markets open are a borrowing and lending market (in which contracts are not state-contingent) and an insurance market for state  $D$ : in this market insurance contracts that will pay off one peso tomorrow if the Democrats win can be bought and sold; the premium (the price of a one-peso contract) is  $p$  pesos, to be paid today. What will the equilibrium interest rate and premium be, how much will each individual save or borrow, and how much insurance will each one buy or sell?

9.5 Andy's income today is \$20 per unit of time (*e.g.*, per hour). If universal health care legislation is passed within the next year, then his income tomorrow will be \$20; but if the legislation fails to pass, his income tomorrow will be only \$10. Beth sells insurance, and her income today is also \$20. If health care legislation is passed, her income tomorrow will be \$10, but if the legislation fails to pass, her income tomorrow will be \$20. Andy's preferences are described by the utility function

$$u^A(x_0, x_H, x_F) = x_0 + 5 \log x_H + 6 \log x_F$$

and Beth's by the function

$$u^B(x_0, x_H, x_F) = x_0 + 10 \log x_H + 3 \log x_F,$$

where  $x_0$  denotes the individual's spending today,  $x_H$  denotes spending tomorrow if the legislation passes, and  $x_F$  denotes spending tomorrow if the legislation fails to pass (all measured in the same units).

- (a) Determine the Pareto efficient allocation(s).
- (b) Determine the Arrow-Debreu allocation(s) and prices.
- (c) Suppose the only markets are spot markets and a credit market. Is the equilibrium allocation Pareto efficient (and how do you know this)? Do not attempt to find the equilibrium interest rate, spot prices, or allocation.

In (d), (e), and (f) you can solve directly, or you can appeal to the complete-markets security pricing formula.

- (d) In the Arrow-Debreu market structure, what is the (implicit) interest rate?
- (e) Suppose the only securities are shares in the firm Gamma Technologies and shares in the firm Delta Insurance. Each share of Gamma will yield \$2 if the legislation passes and \$1 if the legislation fails. Each share of Delta will yield \$1 if the legislation passes and \$2 if it fails. Determine the equilibrium security prices and Andy's and Beth's holdings of securities.
- (f) In the market structure in (e), what portfolio would one have to hold in order to guarantee oneself a return of \$1 tomorrow, whether the legislation passes or not? What would be the cost of the portfolio? What would you say is the interest rate, and why?

9.6 Ann currently has five dollars per hour to spend and Bev has fifteen dollars per hour. In a few years Bev will have to retire; then she will have only four dollars per hour to spend, but then Ann will have sixteen dollars per hour. Both women are making their plans under some uncertainty: they don't know whether the Republicans or the Democrats will be in power when Bev retires. They make their plans based on preferences described by the following utility functions, where  $x_0$  denotes consumption today,  $x_R$  denotes consumption tomorrow (*i.e.*, after Bev retires) in the event that the Republicans are in power, and  $x_D$  denotes consumption tomorrow if the Democrats are in power, and where each  $x_\theta$  is measured in dollars per hour:

$$u^A(x_0, x_R, x_D) = x_0 x^{\frac{1}{2}} x^{\frac{1}{3}} ; \quad \text{i.e.,} \quad MRS^A = \frac{x_0^A}{2x_R^A} \quad \text{and} \quad MRS^A = \frac{x_0^A}{3x_D^A}$$

$$u^B(x_0, x_R, x_D) = x_0 x^{\frac{1}{3}} x^{\frac{1}{2}} ; \quad \text{i.e.,} \quad MRS^B = \frac{x_0^B}{3x_R^B} \quad \text{and} \quad MRS^B = \frac{x_0^B}{2x_D^B}$$

- (a) There is an Arrow-Debreu equilibrium for Ann and Bev in which they each consume ten dollars per hour today. Determine the Arrow-Debreu prices and the women's state-dependent consumptions.
- (b) Suppose that the only market for intertemporal trade is the market for a bond that costs one dollar per hour today and pays off  $1 + r$  dollars per hour tomorrow, no matter which party is in power. Determine the equilibrium rate  $r$  and the state-dependent consumptions, assuming both women behave as price-takers. (*Hint*: This equilibrium also involves Ann and Bev each consuming ten dollars per hour today.)
- (c) Is the equilibrium in (b) Pareto efficient? Explain.
- (d) Suppose there are four securities being traded today: Security #1 pays one dollar per hour if the Republicans are in power and nothing if the Democrats are in power. Security #2 pays one dollar per hour if the Democrats are in power and nothing if it is the Republicans. Security #3 is a bond that pays twelve dollars per hour for certain. Security #4 pays the holder \$60 per hour if the Republicans are in power and requires the holder to pay \$12 per hour if the Democrats are in power. Using  $y_k$  to denote the number of units of Security #k that she buys, write down the constraints that Ann's choices must satisfy.
- (e) Assuming price-taking behavior, what must the equilibrium prices of the four securities in (d) be today (and why), and what will be Ann's and Bev's equilibrium consumption streams?

9.7 The only chips that exist today are X-chips; there are only two X-chips, and Mr. B owns them both. Today's X-chips will perish by tomorrow (any chips not consumed today are wasted), but tomorrow there will again be two X-chips and again Mr. B will own them both. But tomorrow may turn out to be a "high-tech" tomorrow (sometimes referred to as "state  $H$ "), in which case there will also be two Y-chips (which will be extremely powerful), and Mr. A will own both of them. If, alas, tomorrow turns out to be "low-tech" (sometimes referred to as "state  $L$ "), then there will not be any Y-chips. Mr. A and Mr. B make up the entire economy; each has the same preferences, described by the utility function

$$u(x_0, x_L, x_H, y) = x_0 + (1 - \pi) \ln x_L + \pi \ln x_H + 6\pi \ln y,$$

where  $\pi$  is the subjective probability he places on the event that tomorrow will turn out to be high-tech, and where

- $x_0$  denotes his consumption of X-chips today,
- $x_L$  denotes his consumption of X-chips in a low-tech tomorrow,
- $x_H$  denotes his consumption of X-chips in a high-tech tomorrow, and
- $y$  denotes his consumption of Y-chips in a high-tech tomorrow.

Each man believes there is a one-third chance that tomorrow will turn out to be high-tech.

(a) Determine the set of all Pareto optimal allocations. How would your answer be changed if both men were wrong — specifically, what if each believes the probability of a high-tech tomorrow is one-third, but it is actually one-half?

(b) Determine an Arrow-Debreu price-list for contingent claims on all goods.

(c) Now suppose that the only markets that are open today are a spot market for today's X-chips (on which the price is  $p_0 = 1$ ), and two securities markets,  $H$  and  $L$ . A unit of security  $\theta$  ( $\theta = H, L$ ), which can be purchased for  $\psi_\theta$  dollars today, will return one dollar if (and only if) state  $\theta$  occurs. If tomorrow turns out to be high-tech, spot markets for the two kinds of chips will be open, with prices  $q_x$  and  $q_y$ . Of course, if tomorrow turns out to be low-tech, there will be only one good, so no trade will take place in that event. It happens that there is a rational expectations equilibrium in which the spot prices tomorrow are  $(q_x, q_y) = (1, 6)$ . Determine the rest of the equilibrium — *i.e.*, the security prices  $(\psi_H, \psi_L)$ , the quantity of each security purchased or sold by each individual, and the individual consumption levels. Indicate how one can be assured that the equilibrium you have found is indeed an equilibrium.

9.8 Today Anne and Beth are young and productive: they are endowed with, respectively,  $\dot{x}_{A0}$  and  $\dot{x}_{B0}$  units of the economy's all-purpose consumption good, simoleans. Each of them may, with some probability, live long into her retirement years. Denote the four possibilities, or "states," by YY (both survive), NN (neither survives), YN (Anne alone survives), and NY (Beth alone survives). For each of the four states  $\theta$ , let  $\pi_\theta$  denote the probability that the state will occur. Each woman's endowment in her retirement years, if she survives, will be  $\dot{x}_{A1}$  (for Anne) and  $\dot{x}_{B1}$  (for Beth). (Note that each one's old-age endowment is independent of whether the other survives.) Anne's and Beth's preferences for alternative consumption plans are described by the following utility functions, where  $x_{i\theta}$  denotes person  $i$ 's consumption in state  $\theta$  and  $x_{i0}$  denotes person  $i$ 's consumption today:

$$\begin{aligned} u(x_{A0}, x_{AYY}, x_{AYN}) &= \alpha x_{A0} + \pi_{YY} \log x_{AYY} + \pi_{YN} x_{AYN} \\ u(x_{B0}, x_{BYY}, x_{BNY}) &= \alpha x_{B0} + \pi_{YY} \log x_{BYY} + \pi_{NY} x_{BNY} . \end{aligned}$$

Express your answers to the following questions in terms of the parameters that describe the economy – *i.e.*, in terms of  $\alpha$ ,  $\beta$ , the endowments, and the probabilities. Assume that Anne and Beth can exchange goods only among themselves.

- (a) Determine which allocations are Pareto efficient.
- (b) Determine the Arrow-Debreu equilibrium prices and consumptions. (In other words, assume there are markets for deliveries that are contingent on the relevant states occurring; Anne and Beth are the only participants in these markets, and they are price-takers.)
- (c) In (a) and (b) your answers should obviously not have depended upon the probability  $\pi_{NN}$ ; but they should also not have depended upon the probabilities  $\pi_{YN}$  and  $\pi_{NY}$ . Also, even if the women's old-age endowments were not independent of the other's survival, your answer would not have depended upon either  $\dot{x}_{AYN}$  or  $\dot{x}_{BNY}$ . Why are the answers independent of any parameters involving the states YN and NY, and how general is this result?
- (d) Suppose the women's beliefs about their survival probabilities were not the same – *i.e.*, denote Anne's beliefs by  $\pi_\theta^A$  and Beth's by  $\pi_\theta^B$  for each of the four states  $\theta$ , and suppose that  $\pi_{YY}^A \neq \pi_{YY}^B$ . How would this change your answers in (a) and (b)? In particular, would an equilibrium now exist only for very special parameter values; would an equilibrium never exist; would equilibria (when they exist) no longer be Pareto efficient?

9.9 The economy is endowed today with  $\hat{x}_0$  bushels of corn, the only commodity anyone cares about. There will be no endowment tomorrow, but it is possible to grow corn for tomorrow by planting some of today's endowment today. There is some uncertainty about what the growing conditions will be during the intervening period: if conditions turn out to be Good, then each bushel planted will yield  $\alpha_G$  bushels tomorrow; if conditions instead turn out to be Bad, then each bushel planted will yield only  $\alpha_B$  bushels tomorrow. There are  $n$  households in the economy, and each household's preferences are representable by a continuously differentiable utility function  $u_i(x_0^i, x_B^i, x_G^i)$ , where  $x_B^i$  and  $x_G^i$  denote the household's consumption of corn tomorrow in states  $B$  and  $G$  (*i.e.*, under Bad conditions and under Good conditions).

(a) Derive the marginal conditions (expressed in terms of households' marginal rates of substitution) that characterize the interior Pareto efficient allocations. ("Derive" means to show how you obtained the conditions.)

For the remainder of this question, assume that  $\alpha_B = 1$  and  $\alpha_G = 3$ ; that there are only two households, labeled  $a$  and  $b$ ; and that their utility functions are

$$u^a(x_0, x_B, x_G) = x_0 + x_B - \frac{1}{6}x_B^2 + x_G - \frac{1}{36}x_G^2$$

$$u^b(x_0, x_B, x_G) = x_0 + x_B - \frac{1}{6}x_B^2 + x_G - \frac{1}{18}x_G^2 .$$

(b) Determine all the Pareto efficient allocations.

(c) Determine the Arrow-Debreu prices for contingent claims.

(d) Describe an alternative market structure (*i.e.*, alternative to complete contingent claims) for which the rational expectations equilibrium will also be efficient, and explain briefly why efficiency is achieved. What will the interest rate be in this alternative market structure?

9.10 The economy has an endowment of corn. The corn can be allocated to consumption this year and to planting, which will yield corn next year. Each bushel planted this year will yield three bushels next year if temperatures are High during the intervening months, but if the temperatures are Low then each bushel planted this year will yield only two bushels next year. No one looks farther ahead than next year: each consumer has a utility function in which the only arguments are  $x_0$ ,  $x_H$ , and  $x_L$  (consumption this year; consumption next year if temperatures are High; and consumption next year if temperatures are Low). Everyone's utility function is differentiable. Let  $z$  denote the number of bushels that are planted this year, and let  $MRS_H$  and  $MRS_L$  denote an individual's marginal rates of substitution between consumption next year ( $x_H$  or  $x_L$ ) and consumption today ( $x_0$ ).

- (a) Determine the marginal conditions that characterize the Pareto efficient interior allocations.
- (b) Show that the Arrow-Debreu (complete contingent-claims markets) equilibrium is Pareto efficient by showing that it satisfies the marginal conditions you've derived in (a). (It may be helpful to remember that if production has constant returns to scale, then a Walrasian equilibrium must yield zero profit to producers.)

9.11 There are  $n$  consumers, only one commodity, and no production is possible. There is uncertainty about which of two possible events (states of the world) will occur. Let  $\pi_i$  denote consumer  $i$ 's belief about the probability that state 1 will occur (therefore  $1 - \pi_i$  is his belief that state 2 will occur), and let  $x_i(1)$  and  $x_i(2)$  denote consumption by consumer  $i$  in states 1 and 2. Every one of the consumers chooses so as to maximize the expected value (according to his own probability estimate  $\pi_i$ ) of the same function,  $v(z) = z^\alpha$ , where  $z$  denotes his consumption level and where  $0 < \alpha < 1$ . Each consumer's endowment of the commodity is unaffected by the state that occurs.

- (a) Explain how a contingent claims market would operate for this economy. Are there gains to be had by exchange of contracts? Why or why not?
- (b) Derive consumer  $i$ 's demand for  $x_i(2)$  in terms of the price ratio for contracts and  $i$ 's endowment, say  $w_i$ . How will a change in the price ratio or a change in  $w_i$  affect the demand for  $x_i(2)$ ?
- (c) Suppose some  $\pi_i$  increases. How is the Walrasian equilibrium changed?

9.12 Each trader  $i$  is endowed today with  $\dot{x}_0^i$  simoleans. When tomorrow arrives, each trader will again be endowed with simoleans, but his endowment  $\dot{x}_s^i$  will depend upon  $s$ , the state of the world. It's not possible to alter any of the endowments – for example by production, storage, etc. – but the traders can transfer simoleans to one another. Let  $N$  denote the (finite) set of traders  $i$ , and let  $S$  denote the (finite) set of possible states  $s$ . Each trader's preference can be described by a utility function of the form

$$u^i(x_0^i, (x_s^i)_{s \in S}) = x_0^i + \sum_{s \in S} \alpha_s^i \log x_s^i.$$

(a) Write down a maximization problem for which the solutions are the Pareto efficient allocations, and give the first-order marginal conditions (FOMC) that characterize the interior solutions.

(b) Use the FOMC to derive the interior Pareto allocations and the Arrow-Debreu prices as functions of the parameters  $\dot{x}_0$ ,  $(\dot{x}_s)_{s \in S}$ , and  $(A_s)_{s \in S}$ , where  $A_s := \sum_{i \in N} \alpha_s^i$  for each  $s \in S$ .

For the remainder of this problem, let  $N = \{\text{Amy}, \text{Bill}\} = \{A, B\}$ , let  $S = \{\text{High}, \text{Low}\} = \{H, L\}$ , and let

$$\begin{aligned} \alpha_H^A &= 4, & \alpha_L^A &= 2, & \dot{x}_H^A &= 12, & \dot{x}_L^A &= 6, & \dot{x}_0^A &= 10, \\ \alpha_H^B &= 2, & \alpha_L^B &= 4, & \dot{x}_H^B &= 12, & \dot{x}_L^B &= 6, & \dot{x}_0^B &= 10. \end{aligned}$$

(c) Determine the Pareto efficient allocations.

(d) Determine the Arrow-Debreu equilibrium — *i.e.*, the Arrow-Debreu prices and allocation.

(e) Suppose the only market is a credit market, in which Amy and Bill can lend simoleans today in exchange for receiving simoleans tomorrow, or alternatively can borrow simoleans today in exchange for a promise to deliver simoleans tomorrow. Write down Amy's utility-maximization problem and derive the first-order conditions that characterize the decision she will make. Express her FOMC as a condition that relates her marginal rates of substitution to the market rate of interest.

(f) Determine the competitive (price-taking) equilibrium in the credit market in (e).

(g) In addition to the credit market, suppose there is also a security Gamma that returns two simoleans in state  $H$  and three simoleans in state  $L$ . Use the Arrow-Debreu pricing formula to determine the equilibrium price of this security and the equilibrium interest rate. How much will each person borrow or lend, and how much of the security Gamma will each person hold?

9.13 Half the people in the economy are Type A personalities and the other half are Type B. Type A personalities all choose according to the utility function

$$u_A(x_0, x_H, x_L) = x_0 + 5x_H - .3x_H^2 + 3x_L - .3x_L^2$$

and Type B personalities all choose according to the utility function

$$u_B(x_0, x_H, x_L) = x_0 + 5x_H - .4x_H^2 + 3x_L - .2x_L^2,$$

where  $x_0$  represents consumption “today,”  $x_H$  represents consumption “tomorrow” in state H, and  $x_L$  represents consumption “tomorrow” in state L.

Each person is endowed with six units of the good today. Type A people will be endowed with four units tomorrow in state H and only two units in state L; Type B people will be endowed with ten units tomorrow in state H and eight units in state L. Storage of the consumption good from today until tomorrow is not possible. The two states are mutually exclusive and exhaustive.

In your answers, consider only allocations that give all Type A people the same state-contingent consumption bundle and all Type B people the same state-contingent consumption bundle. You can therefore describe an allocation with just the six variables  $x_{A0}, x_{AH}, x_{AL}, x_{B0}, x_{BH}, x_{BL}$ .

- (a) Determine the set of interior Pareto allocations.
- (b) Determine the Arrow-Debreu prices and allocation.
- (c) Assume that the only market is a credit market — a market for borrowing and lending. There are no markets in which one can insure oneself against either of tomorrow’s two possible states. What will be the competitive interest rate, and how much will each person borrow or save?
- (d) In addition to the credit market in (c), suppose there is another market as well, in which one can buy or sell insurance today against the occurrence of state  $H$ . Each unit of insurance that a person purchases is a contract in which the seller of the contract agrees to pay the buyer one unit of consumption tomorrow *if* state  $H$  occurs. Let  $p$  denote the market price of the insurance: the buyer pays the seller  $p$  units of consumption today for each unit of insurance he purchases. Determine the competitive equilibrium prices (i.e., the interest rate and the price  $p$  of insurance) and the equilibrium allocation. How much does each person borrow or save, and how much insurance does each person buy or sell?
- (e) Are the allocations in (c) and (d) Pareto efficient? Explain.

## Other Topics

10.1 Each resident of Porciana can choose to build his house of bricks or straw. If he builds a brick house, his wealth will be  $w_B$ . If he builds a straw house, his wealth will be  $w_S$ , unless there is a hurricane, in which case his wealth will be  $w_H$  if he has built a straw house;  $w_H < w_S$ . Owners of brick houses are unaffected by hurricanes. Everyone in Porciana chooses a house solely on the basis of his von Neumann-Morgenstern utility function for alternative levels of wealth, and everyone is risk averse. Not everyone has the same utility function, however. Everyone knows the probability of a hurricane is  $\pi$  and that  $w_B < (1 - \pi)w_S + \pi w_H$ . Let  $\alpha$  denote the fraction of the population that builds straw houses.

(a) If no hurricane insurance is available, determine the condition that characterizes whether a resident will build a straw house. Can you determine whether  $\alpha = 0$ ,  $\alpha = 1$ , or  $0 < \alpha < 1$ ?

(b) Now assume that insurance firms offer actuarially fair hurricane insurance: a homeowner can buy any amount  $I$  of insurance at a price of  $p$  dollars per unit of  $I$ . Such a policy pays the homeowner  $I$  dollars if there is a hurricane and nothing if there is no hurricane.

(b') How much insurance will each homeowner purchase?

(b'') Determine the condition that characterizes whether a resident will build a straw house. Can you determine whether  $\alpha = 0$ ,  $\alpha = 1$ , or  $0 < \alpha < 1$ ?

(c) Now suppose the government of Porciana institutes a program of hurricane insurance: the government pays  $w_S - w_H$  dollars to all straw house owners when a hurricane occurs, and it finances this program by levying upon all residents an identical lump-sum tax if a hurricane occurs.

(c') How will the pattern of house-building differ (if at all) under this program from (a) and (b)?

(c'') Who (if anyone) is made better off or worse off under this program than under (a) or (b)?

10.2 Show that in a second-price sealed-bid auction, it is a Nash equilibrium for each bidder to bid his or her true value.

10.3 A person (the “seller”) is going to dispose of a single object by auctioning it off. There are only two potential bidders in the auction. The seller is going to use either a First-Price Auction or a Second-Price Auction. In either auction the “winner” (the bidder who will receive the object) will be the one whose bid is the highest; if the two bids are equal, then a fair coin will be flipped to decide the winner; and the “loser” (the one who does not receive the object) will neither pay nor receive anything. In the First-Price Auction the winner will pay to the seller the amount of his bid; in the Second-Price Auction the winner will pay to the seller the amount of the second-highest bid (which, in this two-bidder case, is simply the loser’s bid).

Analyze and compare the two kinds of auction as if each is a two-player game (a game between two bidders; the seller plays no strategic role). Assume the object has the value  $a_i$  to player  $i$ , where  $0 < a_1 \leq a_2$  — i.e., player  $i$  is indifferent between “not receiving the object” and “receiving the object and paying  $a_1$  dollars.” Suppose that only integer bids are allowed. In each auction determine whether either player has a dominant strategy and determine the set of Nash equilibria. How would your analysis and your answers change if the situation were altered in one of the following ways:

- (a) Non-integer bids are allowed.
- (b) The two players are uncertain about one another’s values.
- (c) There are more than two bidders.

10.4 There are two bidders at a first-price sealed-bid auction. Each bidder is risk neutral and each believes the monetary value the other bidder places on the auctioned object was drawn (independently of his own value) according to the uniform distribution on the real interval  $[0, \bar{v}]$ . In addition to the monetary value of the object, each bidder, if he wins, derives further utility equal to  $\alpha$  dollars, simply from being the winning bidder. Derive an equilibrium bid function for this auction.

10.5 A person (called the seller) wishes to sell a valuable painting. There are two potential buyers. The seller has decided to conduct a sealed-bid auction to determine which of the buyers will receive the painting and how much the seller will be paid. The only allowable bids are the non-negative integers: 0, 1, 2, 3, . . . . The painting will go to the highest bidder; if both bids are the same, then a fair coin will be flipped to determine which bidder receives the painting. In any case, the person who receives the painting will pay to the seller the amount of his own bid times one million dollars (i.e., if the receiver of the painting has bid “3” then he will pay three million dollars). Each bidder values the painting at 2 (million dollars), but each is uncertain about the value the opposing bidder places on the painting: each one believes the other bidder values the painting at either 2 (million dollars) or 0, but is not sure which, and he therefore assigns a probability of  $1/2$  to each possibility.

(a) It’s obvious that a bidder who values the painting at 0 will bid 0, but it’s not so obvious what bid will be made by a bidder who values the painting at 2. Determine all the pure-strategy Nash equilibria, and determine whether there are any dominant-strategy equilibria.

(b) Now suppose the seller has decided to use a second-price sealed-bid auction: the high bidder will receive the painting (a fair coin will be flipped if the two bids are equal), but the receiver of the painting will pay to the seller the amount that was bid by the *other* bidder (times one million dollars). What are the pure-strategy Nash equilibria, and are any of them dominant-strategy equilibria?

10.6 In Samuelson's Overlapping Generations model, suppose that there are only two generations alive at any one time; that each person's intertemporal preferences are given by  $u(c_1, c_2) = c_1 c_2$ ; that the population is tripling each generation; that the young are always endowed with 100 units of the non-storable consumption good; and that the elderly are always endowed with 60 units.

(a) Determine the Golden Rule program and determine how much the program requires each young person to transfer to the elderly.

(b) Suppose the stock of money is held today entirely by the elderly, \$120 held by each elderly person. Determine the perfect foresight equilibrium price path that supports the Golden Rule program.

(c) Determine a perfect foresight equilibrium price path in which there is never any trade – everyone's life-cycle consumption stream is simply his endowment stream.

10.7 In the Overlapping Generations model, with two generations alive at each period, let  $\gamma$  denote the growth rate of the population.

(a) Show that if  $\gamma = 0$ , no program that ever gives any younger generation more than it receives in the Golden Rule program could be a Pareto improvement on the Golden Rule allocation.

(b) Show, for any value of  $\gamma$ , that the Golden Rule program is Pareto efficient.