The Core: Bargaining Equilibrium

5.1 There are three consumers; each one's preference is represented by the utility function $u(x, y) = xy$. The first consumer owns the bundle (19,1), the second owns the bundle (1,19), and the third owns the bundle (10,10). Both goods are divisible. Determine the set of all core allocations.

5.2 Bart owns the bundle (16,4); Lisa owns the bundle (8,8); Krusty owns the bundle (4,16). Each one's preference is described by the utility function $u(x, y) = xy$. Consider the proposed allocation in which Bart gets the bundle (10,7); Lisa gets the bundle (7,10); and Krusty gets the bundle (11,11). Determine whether the coalition consisting of Bart and Lisa can improve upon the proposed allocation. Can Bart and Lisa improve on the proposal in which Bart gets (11,7), Lisa gets (7,11), and Krusty gets (10,10)? Is either of the proposals in the core?

5.3 There are two goods and three people in the economy, and all three people have the same utility function: $u(x, y) = xy$. Person #1 is endowed with the bundle $(12, 0)$, and Persons #2 and #3 are each endowed with the bundle $(0, 12)$. In the following cases determine whether the given allocation is in the economy's core. If it is, verify that it is; and if it's not, find an allocation with which some coalition can unilaterally make each of its members better off.

(a) $(x_1, y_1) = (8,8)$, $(x_2, y_2) = (2,8)$, $(x_3, y_3) = (2,8)$

(b) $(x_1, y_1) = (4,8)$, $(x_2, y_2) = (4,8)$, $(x_3, y_3) = (4,8)$

(c) $(x_1, y_1) = (7,14)$, $(x_2, y_2) = (3,6)$, $(x_3, y_3) = (2,4)$

5.4 There are four consumers: $N = \{1, 2, 3, 4\}$. Each consumer's preference is represented by the utility function $u(x, y) = xy$. Consumers #1 and #3 each own one unit of the y-good and none of the x-good; consumers #2 and #4 each own one unit of the x-good and none of the y-good. Both goods are fully divisible.

(a) Suppose a proposed allocation $(\widehat{x}_i, \widehat{y}_i)_{i \in N}$ satisfies $(\widehat{x}_1, \widehat{y}_1) \neq (\widehat{x}_3, \widehat{y}_3)$ or $(\widehat{x}_2, \widehat{y}_2) \neq (\widehat{x}_4, \widehat{y}_4)$, or both, and also (without loss of generality) $u_1(\widehat{x}_1, \widehat{y}_1) \leqq u_3(\widehat{x}_3, \widehat{y}_3)$ and $u_2(\widehat{x}_2, \widehat{y}_2) \leqq u_4(\widehat{x}_4, \widehat{y}_4)$. Prove that the coalition $S = \{1, 2\}$ can improve upon $(\widehat{x}_i, \widehat{y}_i)_{i \in N}$ via the allocation $((\overline{x}_1, \overline{y}_1), (\overline{x}_2, \overline{y}_2))$ to S, where $\overline{x}_1 = \frac{1}{2}(\widehat{x}_1 + \widehat{x}_3)$ and $\overline{x}_2 = \frac{1}{2}(\widehat{x}_2 + \widehat{x}_4)$ and $\overline{y}_1 = \frac{1}{2}(\widehat{y}_1 + \widehat{y}_3)$ and $\overline{y}_2 = \frac{1}{2}(\widehat{y}_2 + \widehat{y}_4)$. Note that what you have proved here is that every core allocation in this economy must satisfy $(\widehat{x}_1, \widehat{y}_1) = (\widehat{x}_3, \widehat{y}_3)$ and $(\widehat{x}_2, \widehat{y}_2) = (\widehat{x}_4, \widehat{y}_4)$.

(b) Determine the set of all core allocations.

5.5 Construct an example to show that if the consumer types are not present in equal numbers, then it is not necessarily true that identical consumers are treated equally in core allocations.

5.6 There are two goods (quantities are denoted by x and y) and no production is possible. Amy, Bev, and Cal all have the exact same preferences for the goods, represented by the utility function $u(x,y) = xy$. Amy owns the bundle $(12,4)$, Bev owns the bundle $(4,4)$, and Cal owns the bundle $(4,12)$. Determine whether each of the following allocations is in the core, and show why your answer is the right one.

(a) $(x_A, y_A) = (8{,}8)$, $(x_B, y_B) = (4{,}4)$, $(x_C, y_C) = (8{,}8)$

(b) $(x_A, y_A) = (4{,}8)$, $(x_B, y_B) = (4{,}4)$, $(x_C, y_C) = (7{,}7)$

5.7 Amy and Bev each own four loaves of bread and no honey. Cal owns eight pounds of honey, but no bread. All three have preferences described by the utility function $u(x,y) = xy$, where x denotes the loaves of bread consumed and y denotes pounds of honey. Determine whether the following allocations are in the core:

(a) Amy: (1,1) Bev: (3,3) Cal: (4,4)

(b) Amy: (2,2) Bev: (2,2) Cal: (4,4)

5.8 Jerry and Elaine have each ordered a large pizza (12 slices each), but have found they have nothing to drink with their pizzas. Kramer has two six-packs of beer (12 bottles), but nothing to eat. They decide to get together for dinner. Each has the same preferences, described by the utility function $u(x,y) = xy$, where x and y denote slices of pizza and bottles of beer.

(a) Derive the utility frontier for each coalition.

(b) Determine whether the following allocation is in the core:

$$(x_J, y_J) = (6,3) \qquad (x_E, y_E) = (4,2) \qquad (x_K, y_K) = (14,7).$$

(c) Kramer is studying economics and recalls that core allocations treat identical individuals identically. What does this “theorem” tell you about the answer to (b)?

5.9 (See Exercises 1.6 and 3.2) The Arrow and Debreu families live next door to one another. Each family has an orange grove that yields 30 oranges per week, and the Arrows also have an apple orchard that yields 30 apples per week. The two households' preferences for oranges (x per week) and apples (y per week) are given by the utility functions

$$u_A(x_A, y_A) = x_A y_A^3 \qquad \text{and} \qquad u_D(x_D, y_D) = 2x_D + y_D.$$

The Arrows and Debreus realize they may be able to make both households better off by trading apples for oranges.

(a) Determine all the Pareto efficient allocations and depict them in an Edgeworth box diagram.

(b) Determine all Walrasian equilibrium price lists and allocations.

(c) Determine all the core allocations.

5.10 Amy has six bottles of beer and Beth has eight bags of peanuts. Amy has no peanuts and Beth has no beer. Amy's and Beth's preferences for beer and peanuts are described by the utility functions

$$u^A(x, y) = y + 12x - x^2 \quad \text{and} \quad u^B(x, y) = y + 12x - \frac{1}{2}x^2,$$

where x denotes bottles of beer consumed and y denotes bags of peanuts consumed. Beer and peanuts are the only goods we will consider, and Amy and Beth are the only traders. Let Z denote the allocation in which Amy consumes $(x, y) = (4, 8)$ and Beth consumes $(x, y) = (2, 0)$.

(a) Is Z Pareto efficient?

(b) Draw an Edgeworth box depicting the set of all Pareto optimal allocations.

(c) Is Z in the core?

(d) Is it a Walrasian equilibrium allocations? If so, give an equilibrium price-list.

(e) Is the Walrasian equilibrium price ratio unique?

5.11 Each person in the following questions cares only about the amounts of the two goods that are allocated to her, and not about how much is allocated to others. Each one's preference is described by the utility function $u(xy) = xy$.

(a) Abby owns the bundle (3,1), Beth owns the bundle (1,3). Determine all the core allocations.

(b) Now suppose that Abby and Beth are joined by a third person, Cathy, who has the same preference as the others, but who owns the bundle (1,1). Determine all the competitive (*i.e.*, Walrasian) equilibria for this three-person economy.

(c) In the three-person economy of (b), determine whether the coalition consisting of Abby and Cathy can unilaterally improve upon the proposed allocation in which Abby and Beth each receive the bundle (2,2) and Cathy receives the bundle (1,1). Prove that your answer is the correct one.

(d) Now suppose the economy consists of 200 people, all of whom have the same preference, described by the utility function $u(x, y) = xy$, and that half the people each own the bundle (3,1) and the other half each own the bundle (1,3). Give as complete a description of the core allocations as you can.

5.12 Ann owns 12 bags of peanuts, but no beer. Bill owns 6 bottles of beer, but no peanuts. Ann and Bob have identical preferences, given by the utility function $u(x, y) = x^2y$, where x and y denote bags of peanuts and bottles of beer consumed, respectively. The Walrasian equilibrium allocation is the one in which Ann consumes the bundle $(\hat{x}_A, \hat{y}_A) = (8, 4)$ and Bob consumes the bundle $(\hat{x}_B, \hat{y}_B) = (4, 2)$. Now suppose that Ann and Bob are joined by Amy and Bill. Amy is identical to Ann (same endowment and same preferences) and Bill is identical to Bob. Note that you don't need to calculate the utility frontiers to solve this problem; those calculations are a bit complicated.

(a) Now that all four people are available to trade with one another, the only Walrasian allocation is the one that gives *both* Ann and Amy the bundle $(\hat{x}_A, \hat{y}_A)$ and both Bob and Bill the bundle $(\hat{x}_B, \hat{y}_B)$. Prove that this allocation is in the core.

(b) Now consider the bundles $(\bar{x}_A, \bar{y}_A) = (4, 2)$ and $(\bar{x}_B, \bar{y}_B) = (8, 4)$. When Ann and Bob are the only two people who are going to exchange beer for peanuts, it is easy to see that this allocation is in the core. Determine whether the allocation that gives the bundle $(\bar{x}_A, \bar{y}_A)$ to both Ann and Amy, and the bundle $(\bar{x}_B, \bar{y}_B)$ to both Bob and Bill, is a core allocation when all four of them can exchange beer and peanuts with one another.

5.13 Amy, Beth, and Carol each own orange trees, and they each like to eat oranges and drink orange juice. Their preferences and their trees' daily yield of oranges are as follows, where x_i denotes i's consumption of oranges per day (either as oranges or juice) and y_i denotes i's spending (in pennies per day) on all other goods:

	Yield	Utility Function
Amy	10	$y_A + 60x_A - 3x_A^2$
Beth	15	$y_B + 60x_B - (3/2)x_B^2$
Carol	5	$y_C + 60x_C - x_C^2$

Each woman has been consuming only the oranges from her own tree. Find a Pareto improvement that is also in the core. Explain how you know that your proposed allocation is both a Pareto improvement and in the core. (Assume that each woman's daily income $\mathring{y}_i$ is at least 1000.)

5.14 (See Exercise 4.5) Andy, Bob, and Cathy each have the same preferences for wine and grapes, described by the utility function $u(x, y) = xy$, where x and y denote an individual's consumption of wine (x gallons) and grapes (y bushels). Grapes can be turned into wine; it takes three bushels of grapes to produce each gallon of wine. This production process is available to everyone – i.e., everyone has the ability to produce wine from grapes at this rate.

(a) Suppose Andy and Bob each own 12 bushels of grapes and Cathy owns 24 bushels of grapes. No one owns any wine. Determine the Walrasian equilibrium prices, production levels, and consumption bundles.

In parts (b) and (c), either prove that the proposed allocation is in the core (by showing that no coalition can do better for itself), or prove that it is not (by showing that some coalition can do better for itself).

(b) With the endowments in (a), determine whether the following allocation is in the core:

Andy: (1,3) Bob: (3,9) Cathy: (4,12)

(c) Now suppose that Andy and Bob each own 4 gallons of wine, but no grapes, and Cathy owns 24 bushels of grapes, but no wine. (It is not possible, of course, to turn wine into grapes.) Determine whether the following allocation is in the core:

Andy: (2,6) Bob: (4,0) Cathy: (4,12)

5.15 Acca and Bayab are tiny islands in the gulf of Ababa. Anna is the sole resident and owner of Acca, and Bob is the sole resident and owner of Bayab. Anna and Bob consume only apples and bananas, and each has the same preference ordering, represented by the utility function $u(x, y) = xy$, where x and y denote the consumer's daily consumption of apples and bananas. Only apples are grown on Acca, where the yield is 20 apples per day, and only bananas are grown on Bayab, where the yield is 10 bananas per day.

Anna and Bob have been trading with one another (by boat) for years: Anna gives Bob 12 apples every day in return for 4 bananas. Call this situation and the resulting consumption allocation SQ, for "status quo."

(a) Is SQ Pareto Optimal? Is it in the core? Verify your answers.

(b) Is SQ a Walrasian equilibrium allocation? If so, determine the associated prices; if not, could both Anna and Bob be made better off by organizing their exchanges in terms of markets and prices?

Now suppose that Anna discovers a technology, called the Alpha technology, with which she can transform apples into bananas, at the rate of two apples for each banana obtained. Answer (c), (d), and (e) assuming that Anna is the sole owner of the Alpha technology.

(c) Now is SQ Pareto optimal? If so, verify it; if not, find a Pareto improvement

(d) How does the discovery of the new technology affect the set of core allocations? Is SQ in the core now?

(e) How does the discovery of the new technology affect the set of Walrasian allocations and prices?

(f) Now suppose it is not Anna who discovers a technology, but it is Bob. Bob discovers the Beta technology, with which he can transform two bananas into one apple as often as he likes. Then is SQ in the core? Will the set of Walrasian allocations and/or prices be affected (as compared to the original, no-technology situation)?

(g) Now suppose that both technologies have been discovered. How will the set of Walrasian allocations and prices be affected by the ownership of the technologies? In particular, compare the Walrasian equilibria when Anna owns Alpha and Bob owns Beta to when Anna owns Beta and Bob owns Alpha. What would happen to the Walrasian outcomes if one of the people owned both technologies?

5.16 (See Exercises 1.2 and 3.6) Ann and Bob each own 10 bottles of beer. Ann owns 20 bags of peanuts and Bob owns no peanuts. There are no other people and no other goods in the economy, and no production of either good is possible. Using x to denote bottles of beer and y to denote bags of peanuts, Ann's and Bob's preferences are described by the following utility functions:

$$u_A(x_A, y_A) = x_A y_A^4 \qquad \text{and} \qquad u_B(x_B, y_B) = 2x_B + y_B.$$

Note that their MRS schedules are $MRS_A = y_A/4x_A$ and $MRS_B = 2$.

(a) Determine all Walrasian equilibrium price lists and allocations.

(b) Determine all core allocations.

5.17 Abby's and Beth's preferences are both described by the utility function $u(x, y) = xy$. Abby owns the bundle (4,1), Beth owns the bundle (1,4).

(a) Determine the Walrasian equilibrium allocation and prices. You needn't do this by deriving the equilibrium, but you should verify that what you have is an equilibrium.

(b) Determine the set of core allocations.

Now suppose that Abby and Beth are joined by a third person, Cathy, who has the same preference as the others, but who owns the bundle (2,2).

(c) Determine the Walrasian equilibrium for this three-person economy. Again, you only have to verify that the equilibrium you've identified is actually an equilibrium.

(d) Determine the set of core allocations in the three-person economy.

5.18 We begin with a 2×2 "Edgeworth Box" exchange economy: each consumer has the same preference, described by the utility function $u(x, y) = xy$; Consumer 1 owns the bundle $(\mathring{x}_1, \mathring{y}_1) = (15, 30)$; and Consumer 2 owns the bundle $(\mathring{x}_2, \mathring{y}_2) = (75, 30)$.

(a) Verify that there is a unique Walrasian (competitive) equilibrium, in which the price ratio is $p_x/p_y = 2/3$ and the consumption bundles are $(x_1, y_1) = (30, 20)$ and $(x_2, y_2) = (60, 40)$.

(b) Verify that the Pareto allocations are the ones that allocate the entire resource endowment of $(\mathring{x}, \mathring{y}) = (90, 60)$ and satisfy $y_1/x_1 = y_2/x_2 = 2/3$.

(c) In the Edgeworth Box draw the competitive allocation, the Pareto allocations, and each consumer's budget constraint. Draw each consumer's indifference curve containing his initial bundle and indicate the core allocations in the diagram.

(d) Verify that the Pareto allocations for which $x_1 < \sqrt{675}$ are not in the core. Note that $\sqrt{675}$ is approximately 26. Similarly, the Pareto allocations for which $x_2 < \sqrt{3375} \approx 58.1$ are not in the core.

(e) Consider a proposed allocation $(\hat{x}_1, \hat{y}_1) = (27, 18)$ and $(\hat{x}_2, \hat{y}_2) = (63, 42)$. Note that each consumer's marginal rate of substitution at the proposal is 2/3. Verify that the proposal is in the core. Verify that the "trading ratio" τ defined by the proposal is $\tau = 1$. As in our lecture notes on the Debreu-Scarf Theorem, use the "shrinkage factor" $\lambda_1 = 2/3$ and the "expansion factor" $\lambda_2 = 4/3$ to verify that a coalition of just two "Type 1" consumers and one "Type 2" consumer can unilaterally allocate their initial bundles to make all three of them better off than in the proposal. Therefore the proposal is not in the core if there are two or more consumers of each type.

(f) Now consider the proposal $(\hat{x}_1, \hat{y}_1) = (28\frac{1}{2}, 19)$ and $(\hat{x}_2, \hat{y}_2) = (61\frac{1}{2}, 41)$, and use the same λ_1 and λ_2 as in (e) to establish that this proposal too is not in the core if there are two or more consumers of each type.

(g) Now consider the proposal $(\hat{x}_1, \hat{y}_1) = (29, 19\frac{1}{3})$ and $(\hat{x}_2, \hat{y}_2) = (61, 40\frac{2}{3})$, and use the factors $\lambda_1 = 4/5$ and $\lambda_2 = 6/5$ to establish that this proposal is not in the core if there are three or more consumers of each type.