Solution to Dutta's Exercise #4.2

We want to show that a player has an undominated strategy – i.e., that he has some strategy that is not dominated by any other strategy. The way we'll do this is to suppose that every one of his strategies *is* dominated by some other strategy, and show that that can't happen.

Let's suppose the player has M strategies altogether. Suppose all M of these strategies are dominated: for each strategy *s*, there is another strategy *s*' that dominates *s*. Start with any one of the player's strategies, and call it Strategy 1. It's dominated by some other strategy; call that Strategy 2. Strategy 2 is dominated by some strategy; call that one Strategy 3. And so on. *After at most M steps, we'll have run out of new strategies that could dominate; so the strategy that dominates the one from the previous step will have to be one of the strategies we already considered*. In other words, we'll have a "cycle" of domination. But then, because of the **Proposition** proved below, each strategy in the cycle would dominate itself. But no strategy can ever dominate itself, because that would require a payoff in some cell to be larger than itself. So we have established that indeed the player's strategies cannot all be dominated: assuming they're all dominated leads to a conclusion that can't be true.

The following proposition was an essential step in the above argument (or "proof"):

Proposition: If Strategy *A* dominates Strategy *B*, and Strategy *B* dominates Strategy *C*, then Strategy *A* has to dominate Strategy *C*.

Proof:

Against every configuration (or "profile") of strategies by the other players, *A* yields our player at least as large a payoff as *B*, and *B* yields at least as high a payoff as *C* (because *A* dominates *B*, and *B* dominates *C*). Moreover, there is some profile against which *A* yields a *strictly* higher payoff than *B*, and also a profile against which *B* yields a strictly higher payoff than *C*; against each of these two profiles, then, *A* must yield a strictly higher payoff than *C*.

Exercise: Put each statement in the proposition's proof into symbolic form. For example, the first statement is

$$\text{For every profile } s_{-i}\text{:}\quad \pi_i(A, s_{-i}) \geq \pi_i(B, s_{-i}).$$

Exercise: Dutta specified that the player has only a finite set of strategies – we said M of them, for example. Why did Dutta do this? Presumably because a player might not actually have an undominated strategy if he has an infinite number of strategies. Make up a two-player game in which Andy's strategies are the positive integers – i.e., his strategies are to choose 1, or 2, or 3, or any other positive integer, no matter how large. Becky, on the other hand, has only two strategies: she can choose either "Left" or "Right." Make up payoffs for this game which have the property that Andy has no undominated strategy. (Hint: When a player has an infinite number of strategies, the italicized phrase above need not be true – we can keep on finding new dominating strategies forever.)