Exercise Set #9

Part I (Nim)

The game Nim is defined as follows. Several rows of matchsticks are laid on the table. There can be any number of rows, and any number of matchsticks in any of the rows. We will analyze only Nim games that have two rows, and we will label them Row A and Row B. There are two players, whose moves are governed by the following simple rules: A player, when it is her turn to move, may pick up any number of the matchsticks that still remain in any *one* of the rows. Then it is the other player's turn to move. The winner is the player who picks up the last matchstick -- i.e., the player who "clears the board." Note that a draw is not possible under the rules.

To simplify the analysis of Nim, note that it never matters, strategically, *which* matchsticks a player picks up. That is, if a player picks up, say, k matchsticks from Row A, it makes no difference *which* k matchsticks she picks up. Thus, denote moves by the number of matchsticks picked up and the row from which they are picked up. For example, the move "pick up two matchsticks from Row A" is denoted A2. (Equivalently, you can denote a move by the row and the number of matchsticks the player *leaves* in the row. It doesn’t matter which way you do it.)

1. Is Nim a game of perfect information? Explain.

2. Write down the extensive form game tree and determine the backward-induction equilibria for the following versions of Nim, each of which has two rows, Row A and Row B:

(a) Row A has one matchstick, Row B has two matchsticks.

(b) Each row has two matchsticks. Since it is immaterial which row's matchsticks are taken on the first move, assume that they are drawn from Row A; this cuts in half the tree you will have to work with. (Note, though, that it is not immaterial *how many* are taken from Row A.)

(c) Row A has one matchstick and Row B has three.

3. Write down the normal form for the Nim game in (a) and determine its Nash equilibria. What is the game’s solution according to iterated elimination of dominated strategies? How many rows and columns are there in the normal form of the game in (b)? In (c)?

4. Suppose we reverse the definition of the winner -- i.e., we say that the one who picks up the last matchstick is the *loser*. How does this change the answers to (1), (2), and (3)? How does this affect the winner in each case -- i.e., in each case, is it now the opposite player who has a strategy that ensures a win, or the same player as before?

Part II (Poker)

Here is a simple poker game. Player 1 antes two dollars and Player 2 antes one dollar. Then Player 1 is dealt a single card from a deck that consists of half Aces and half Kings, and he is allowed to observe the card he's been dealt. If Player 1 has been dealt an Ace, he *must* place an additional three dollars into the pot, but without telling Player 2 what his card is. If he has been dealt a King he can choose to Bet (placing three dollars into the pot) or to Fold. Player 2 observes whether Player 1 has Bet or Folded. If Player 1 Folded, the game is over and Player 2 wins the pot (thus winning Player 1's ante, two dollars). But if Player 1 has Bet, then Player 2 must choose either to Call him (placing three dollars into the pot) or to Fold. If Player 2 Folds, then Player 1 wins the pot (thus winning Player 2's ante, one dollar). If Player 2 Calls, then Player 1 shows his card; if it is an Ace, Player 1 wins the pot (thus winning four dollars from Player 2), and if the card is a King, Player 2 wins the pot (thus winning five dollars from Player 1).

1. Write down the extensive and normal forms of this game and determine the Nash equilibrium and the game's value. **Answer:** Player 1 Bets ("bluffs") half the time when he has a King. Player 2 Calls half the time when he has the chance to (i.e., after Player 1 has bet). Player 1 has an expected value of 1/4 dollar when they use these strategies.

2. Is this a game of perfect information? Explain.

3. Poker is a signaling game in which there is asymmetric information: the hand a player is dealt is known only to him, not to the other players, and a player's actions (his bets) are "signals" that convey information about the hand he has been dealt. Use Bayes' Rule to determine the conditional probabilities Player 2 should hold about Player 1's card after he has seen whether Player 1 has bet: i.e., determine the Pr(s | d) for s = A and for s = K, and for d = Bet and for d = Fold. In order to determine these probabilities, assume that Player 1 is playing his minimax mixture (i.e., his part of the Nash equilibrium). Then determine the expected payoff to Player 2 (using these "updated" probabilities) from Calling and from Folding. If mixing between Call and Fold is a best response for Player 2 to what Player 1 is doing, then these two expected payoffs have to be the same. Are they?

4. Instead of the players having to bet \$3 when they bet, we could change the rules to make this any amount we like. Let's denote the amount a player has to bet to remain in the pot by **b**. Determine what **b** would have to be in order to make this a fair game – i.e., in order that the value of the game is zero to each player. (We are still using the rule that Player 1 antes \$2 and Player 2 antes \$1.) In order to solve this problem you will have to do a little algebra, or else you will have to put together a spreadsheet to help you out.