1. Numero Uno and Dos Hermanos both sell identical cases of bottled water. Market demand is given by \( p = 100 - 5Q \), where \( Q \) is the total cases of water consumers will purchase if the price is \( p \) dollars per case. It costs Numero Uno \( 10q_1 \) dollars to produce \( q_1 \) cases of water; it costs Dos Hermanos \( 40q_2 \) dollars to produce \( q_2 \) cases. Assume throughout that each firm attempts to maximize its profit.

(a) Suppose the firms each take as given the production level of the other firm. Determine the firms’ reaction functions and draw both reaction curves in a single diagram. Determine geometrically and analytically the Cournot-Nash equilibrium production levels and the market price. Determine each firm’s profit, the consumer surplus, and the total surplus.

(b) Now suppose a third firm, Tres Abuelas, enters the market; it costs Tres Abuelas \( 10q_3 \) to produce \( q_3 \) cases of water. Determine the Cournot-Nash equilibrium production levels by each firm and the market price. Determine each firm’s profit and the consumer surplus.

(c) Now suppose that each of the three firms takes the market price as given instead of taking the other firms’ production levels as given. Determine the equilibrium price and the production levels of each firm. Determine each firm’s profit, the consumer surplus, and the total surplus.

(d) Suppose the market demand function given above is the sum of the demands of twenty identical consumers. Determine a utility function that represents the preference of the typical consumer. Show that the typical consumer’s utility is greater in (c) than in (a).

(e) Now suppose there is some differentiation in the firms’ products, so that the firms may be able to all sell positive quantities at different prices. If you were given the demand functions for each firm’s product, you could determine both the Cournot equilibrium (“the firms compete in quantities”) and the Bertrand equilibrium (“the firms compete in prices”). The two equilibria will be different: \( i.e. \), the production levels and the prices the firms charge will be different in the two equilibria. But this seems like a paradox: when a firm takes its competitors’ actions as given, that generates a residual demand curve for the firm’s product; it’s then immaterial whether the firm is choosing its price (and the residual demand curve determines the quantity it will sell) or is choosing its production level (and the residual demand curve determines the price it will receive when selling that quantity). It seems as if it should make no difference whether the firms are “choosing quantities” or “choosing prices.” Explain why this seeming paradox – that the Cournot and Bertrand equilibria differ – is not a paradox.
2. In Problem #1 Numero Uno and Tres Abuelas have an incentive to collude: if each restricts its production to 4\(\frac{1}{2}\) units, each earns about $200 profit. (Assume for this problem that Dos Hermanos does not compete: it chooses not to produce.) If one of the two firms “cheats” on the collusive agreement and produces 6 units, while the other firm adheres to the agreement and produces 4\(\frac{1}{2}\) units, then the cheating firm will earn about $225 and the adhering firm will earn only about $170. If each firm cheats, each earns about $180. (These profit numbers are rounded off from the exact numbers to make the arithmetic in this problem easier.) You are asked to analyze this \(2 \times 2\) game.

(a) Write down the bimatrix payoff table for this game. What is the Nash equilibrium of the game?

(b) We often don’t see Nash equilibria attained until participants have played a game a number of times, so that each player has a pretty accurate expectation of the actions the other players will choose. Thus, if a game is to be played only once, the Nash equilibrium may not be a good prediction. If Numero Uno and Tres Abuelas play the \(2 \times 2\) game described above just a single time, how would their likely uncertainty about one another’s actions affect your prediction about the outcome?

(c) Now suppose the two firms know they are going to compete against one another in this game exactly twice, and that the outcome of the first encounter will be known to each firm prior to the second encounter. Numero Uno is trying to determine whether it would be better to simply cheat at each of the two “stages” of play or to instead adhere at the first stage (producing 4\(\frac{1}{2}\) units) and then cheat at the second and final stage. Let \(p\) denote Numero Uno’s estimate of the probability Tres Abuelas will adhere at the first stage and then copy Numero Uno’s first-stage action at the second stage (i.e., that Tres Abuelas will play “tit for tat”). Numero Uno believes that with probability \(1 - p\) Tres Abuelas will simply cheat at both stages. Determine how large \(p\) would have to be in order for Numero Uno to adhere to the agreement at the first stage.

(d) Generalize (informally) from your results in (a), (b), and (c) to explain how we might expect the firms to behave if they believe they will be engaging in this same strategic situation repeatedly over a long period of time.
3. Arnie, Ben, and Chris own a golf course. The quality of the greens depends on the number of dollars they spend on maintenance. Let $x$ denote the quality of the greens and let $y_i$ denote the number of dollars player $i$ spends on other goods. Player $i$’s preference is described by the utility function $u_i(x, y_i)$. The cost of maintaining the greens at level $x$ is $C(x)$ dollars.

(a) Derive the Samuelson marginal condition that characterizes the interior Pareto allocations.

From now on assume that $C(x) = 3x$ and that the utility functions are $u_i(x, y_i) = y_i + \alpha_i \ln x$, where $\alpha_1 = 1$, $\alpha_2 = 2$, and $\alpha_3 = 3$, and assume that each player’s wealth is $100.

(b) Determine the set of Pareto allocations.

(c) Determine the Lindahl prices and the Lindahl allocation.

(d) Suppose the level of maintenance (i.e., $x$, the quality of the greens) is determined via voluntary contributions. Let $t_i$ denote the amount each player contributes. Determine the Nash equilibrium level $x$ of maintenance and the associated contribution by each player.

(e) Now suppose that the players have agreed to use the following method each week to determine that week’s level of maintenance $x$ and the payments $t_1$, $t_2$, and $t_3$ each player will make: Each will place a request $r_i$ with the maintenance company; the company is then authorized to maintain the greens at level $x = r_1 + r_2 + r_3$ and to charge the players the amounts

$$t_1 = (1 + r_2 - r_3)x \quad t_2 = (1 + r_3 - r_1)x \quad t_3 = (1 + r_1 - r_2)x.$$

Note that $t_1 + t_2 + t_3$ will always be equal to $3x$. Verify that the Nash equilibrium $(r_1, r_2, r_3)$ yields the Lindahl allocation.