Chapter 12

General Equilibrium with Uncertainty

12.1 Introduction

In this chapter we will make a brief foray into the theory of general equilibrium with uncertainty. If you remember the discussion of Chapter 2, you will recall that in general equilibrium theory, a commodity is defined by (1) its physical description, (2) its location, (3) the time at which it is available, and (4) the state of the world in which it is available. Consequently, in most of this chapter we are *specializing* the theory which we have been studying; putting more structure into the model in order to account for the effects of uncertainty. Of course, when one delves more deeply into this theory, questions arise which did not appear to be relevant in our earlier studies. Moreover, if we were to pursue the subject to its current frontiers, we would find that new theoretical concepts and tools are needed to provide answers for these questions. However, in the interests of practicality, we will only attempt to provide a 'bare bones' introduction to this theory. Fortunately, in even this cursory introduction to the topic, we will find that some interesting issues and applications can be discussed. We will begin our discussion with what is known as the 'Arrow-Debreu Contingent Commodities Model.'

12.2 Arrow-Debreu Contingent Commodities

The crux of this model is that we suppose that there are two periods, $t = 0, 1$. At $t = 0$, it is supposed that we can set forth all possibilities for the state of the world at the second date, $t = 1$. We assume that there is a finite set, S, of such possible states, and we will also write $S = \#S$; denoting states by lower case 's, s',' etc. Each 'state' is a complete description of the world, and in this theory, we suppose that every agent will know which state, $s \in S$ has occurred once we reach $t = 1$.

We will suppose that there are G physically distinguishable commodities (which, in principle could also be distinguished by location), so that 'n,' the dimension of our commodity space becomes:

$$n = S \cdot G.$$

Commodity bundles then take the form:

$$\boldsymbol{x} = (x_{11}, \ldots, x_{1G}, x_{21}, \ldots, x_{sg}, \ldots, x_{SG}),$$

which is understood to be an entitlement to receive the commodity bundle:

$$\boldsymbol{x}_s = (x_{s1}, \ldots, x_{sG}),$$

if state s occurs. Thus 'x_{sg}' denotes the amount of commodity g to be received (or supplied, if $x_{sg} < 0$) if state s occurs.

In further specifying the economy, we will depart from our previous notation to denote consumer i's resource endowment, by '$\boldsymbol{\omega}_i$,' which now takes the form:

$$\boldsymbol{\omega}_i = (\omega_{i11}, \ldots, \omega_{i1G}, \omega_{i21}, \ldots, \omega_{isg}, \ldots, \omega_{iSG}); \tag{12.1}$$

that is, 'ω_{isg}' denotes consumer i's endowment of the g^{th} commodity if state s occurs. Fortunately, we will rarely have to write out the full vector as we've done in (12.1), above. Defining

$$\boldsymbol{\omega}_{is} = (\omega_{is1}, \ldots \omega_{isG}) \quad \text{for } s = 1, \ldots, S;$$

that is, letting '$\boldsymbol{\omega}_{is}$' denote i's endowment if state s occurs, the finest detail we will usually write out is:

$$\boldsymbol{\omega}_i = (\boldsymbol{\omega}_{i1}, \ldots, \boldsymbol{\omega}_{is}, \ldots, \boldsymbol{\omega}_{iS}).$$

We suppose also that the consumer's preferences describe a weak order over X_i, denoted by '$\succsim_i$. Furthermore, we denote the i^{th} consumer's consumption bundle, contingent upon the occurrence of state s by '$\boldsymbol{x}_{is}$;' so that we can write:

$$\boldsymbol{x}_i = (\boldsymbol{x}_{i1}, \ldots, \boldsymbol{x}_{is}, \ldots, \boldsymbol{x}_{iS}).$$

Similarly, we will let '$Y_k \subseteq \mathbb{R}^n$' denote the feasible production plans for the k^{th} firm, and we will use the generic notation:

$$\boldsymbol{y}_k = (\boldsymbol{y}_{k1}, \ldots, \boldsymbol{y}_{ks}, \ldots, \boldsymbol{y}_{kS}), \tag{12.2}$$

to denote elements of Y_k, where '$\boldsymbol{y}_{ks}$' denotes the production vector of the firm, contingent upon the occurence of state s. We then complete the model, departing from our previous notation,[1] by letting 'θ_{ik}' denote the i^{th} consumer's share in the k^{th} firm's profits.

We will have to be a bit careful in dealing with individual consumption and production sets. One is tempted, for example to express the i^{th} consumer's consumption set as:

$$X_i = \prod_{s=1}^{S} X_{is}, \tag{12.3}$$

where 'X_{is}' denotes the i^{th} consumer's feasible consumption set if state s occurs; with similar specifications for the firms' production sets. That this will not quite do is perhaps best illustrated by considering the following production example, which is inspired by Mas-Colell, Whinston, and Green [1995, Example 19.B.2, p. 689].

[1] This change is made in order that the i^{th} consumer's shares not be confused with the i^{th} state of the world.

12.1 Example. Suppose there are two states, s_1 and s_2, representing good and bad weather. There are two physical commodities: seeds ($g = 1$) and crops ($g = 2$). In this case, the elements of Y_k are four-dimensional vectors. Assume that seeds must be planted before the resolution of the uncertainty about the weather and that if the weather is good, the firm's production possibilities are given by:

$$Y_{k1} = \{\boldsymbol{y}_{k1} \in \mathbb{R}^2 \mid y_{k12} \geq 0 \ \& \ 2y_{k11} + y_{k12} \leq 0\};$$

whereas in bad weather, production is given by:

$$Y_{k2} = \{\boldsymbol{y}_{k2} \in \mathbb{R}^2 \mid y_{k22} \geq 0 \ \& \ y_{k21} + 2y_{k22} \leq 0\}.$$

Recalling our assumption that the seed must be planted before the resolution of uncertainty, we see that we can represent the firm's production set as:

$$Y_k = \{\boldsymbol{y}_k \in Y_{k1} \times Y_{k2} \mid y_{k11} = y_{k21}\}. \tag{12.4}$$

Thus, for example, the production vector:

$$\boldsymbol{y}_k = (y_{k11}, y_{k12}, y_{k21}, y_{k22}) = (-2, 4, -2, 1),$$

is a feasible plan; whereas neither of the production plans:

$$\boldsymbol{y}_k = (-4, 8, -2, 1) \text{ and } \boldsymbol{y}' = (-2, 4, 0, 0),$$

is feasible. □

While the above example deals with a production set, the difficulty applies equally to consumption sets; after all, someone has to plant the seeds, and this labor must also be undertaken before the resolution of uncertainty. In order to allow for this fact, while yet being able to assume on some occasions that consumers' preferences are weakly separable over states, we will assume that for each consumer there exist sets:

$$X_{is} \subseteq \mathbb{R}^G,$$

representing the consumer's feasible consumption possibilities if state s occurs (for $s = 1, \ldots, S$), and a set $\widehat{G}_i$ (presumably a proper subset of G), such that:

$$X_i = \left\{\boldsymbol{x}_i \in \prod_{s=1}^{S} X_{is} \mid (\forall g \in \widehat{G}_i)\colon x_{i1g} = x_{i2g} = \cdots = x_{iSg}\right\} \tag{12.5}$$

Thus, with this specification, one can make sense of the following example.

12.2 Example. Suppose that, for a given consumer, i, there exist S utility functions:

$$u_{is}\colon \mathbb{R}^G \to \mathbb{R},$$

such that:

$$\boldsymbol{x}_i \succsim_i \boldsymbol{x}'_i \iff \left[\sum_{s\in S} \pi_{is} u_{is}(\boldsymbol{x}_{is}) \geq \sum_{s\in S} \pi_{is} u_{is}(\boldsymbol{x}'_{is})\right]. \tag{12.6}$$

where 'π_{is}' denotes i's subjective (or objective) probability of the occurrence of state s. Notice that, even though we have state-dependent utility here, preferences are weakly separable on X_{is}, for each state, s. □

We will not need to assume much about the form of the firms' production sets,[2] we will simply suppose that the k^{th} firm's technological production possibilities are given by a production set $Y_k \subseteq \mathbb{R}^n$.

We will make use of the following definition of feasible allocations for the economy.

12.3 Definition. We will say that an allocation, $(\langle \mathbf{x}_i^* \rangle, \langle \mathbf{y}_k^* \rangle) \in \mathbb{R}^{(m+\ell)n}$ is **feasible for** $\mathcal{E}$ iff:

$$\begin{aligned} \mathbf{x}_i^* &\in X_i \quad \text{for } i = 1, \ldots, m, \\ \mathbf{y}_k^* &\in Y_k \quad \text{for } k = 1, \ldots, \ell, \end{aligned} \tag{12.7}$$

and:

$$\sum_{i \in M} \mathbf{x}_i^* = \sum_{i \in M} \boldsymbol{\omega}_i + \sum_{k \in L} \mathbf{y}_k^*. \tag{12.8}$$

Now, at this point, you may be saying, or thinking, "Hold on! That's exactly the definition of a feasible allocation which was presented in Chapter 7!" And in saying this you are absolutely right! All we have done here so far is to present a somewhat more detailed and specialized specification of what the commodity space is. However, notice that equation (12.8) of the above definition implies that:

$$\sum_{i \in M} \mathbf{x}_{is}^* = \sum_{i \in M} \boldsymbol{\omega}_{is} + \sum_{k \in L} \mathbf{y}_{ks}^* \quad \text{for } s = 1, \ldots, S; \tag{12.9}$$

so that in each state, consumption equals net supply.

To continue our interpretation of the Arrow-Debreu Contingent Commodities Model, the interpretation of the equilibrium which we are now going to discuss is that at time $t = 0$ there is a futures market for each contingent commodity. Equilibrium will require that supply equals demand for each contingent commodity.

12.4 Definition. A system of prices, $\mathbf{p}^* = (p_{11}^*, \ldots, p_{SG}^*) \in \mathbb{R}^n$ and an allocation, $(\langle \mathbf{x}_i^* \rangle, \langle \mathbf{y}_k^* \rangle)$ will be said to be an **Arrow-Debreu equilibrium** iff:

1. $(\langle \mathbf{x}_i^* \rangle, \langle \mathbf{y}_k^* \rangle)$ is a feasible allocation,
2. for every $k \in L$, $\mathbf{y}_k^*$ satisfies:

$$(\forall \mathbf{y}_k \in Y_k): \mathbf{p}^* \cdot \mathbf{y}_k^* \geq \mathbf{p}^* \cdot \mathbf{y}_k,$$

and:

3. for every $i \in M$:

$$\begin{aligned} &\mathbf{p}^* \cdot \mathbf{x}_i^* \leq \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{k \in L} \theta_{ik} \mathbf{p}^* \cdot \mathbf{y}_k^*, \text{ and :} \\ &(\forall \mathbf{x}_i \in X_i): \mathbf{x}_i \succ_i \mathbf{x}_i^* \Rightarrow \mathbf{p}^* \cdot \mathbf{x}_i > \mathbf{p}^* \cdot \boldsymbol{\omega}_i + \sum_{k \in L} \theta_{ik} \mathbf{p}^* \cdot \mathbf{y}_k^*. \end{aligned} \tag{12.10}$$

[2] Other than to keep in mind the fact that it is probably inappropriate to suppose that they can be written as a cartesian product of state-specific production sets.

Once again, the definition is formally identical to that presented in Chapter 7; the only difference is in the interpretation. The beauty of the situation, however, is that we can immediately deduce some important results. In particular, we can see that if $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is an Arrow-Debreu equilibrium then $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle)$ must be Pareto efficient; at least in terms of the *ex ante* consumer preferences, $\succsim_i$.

The following example may help you to get a better 'feel' for the model and the meaning of the definition of competitive equilibrium being used here.

12.5 Example. Suppose $\mathcal{E}$ is an exchange economy with $m = 2 = S$, and $G = 1$; that is we have two consumers, one physically distinguishable commodity, and two states of the world to consider. We will also suppose that the i^{th} consumer has a twice-differentiable Bernoullian utility function, $u_i \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $x \in \mathbb{R}_+$:

$$u_i'(x) > 0 \text{ and } u_i''(x) < 0;$$

so that u_i is strictly increasing and strictly concave. If $\boldsymbol{x}_i = (x_{i1}, x_{i2})$ and $\boldsymbol{x}_i' = (x_{i1}', x_{i2}')$ are two commodity bundles in $X_i = \mathbb{R}_+^2$, consumer i will prefer $\boldsymbol{x}_i$ to $\boldsymbol{x}_i'$ if, and only if:

$$U_i(\boldsymbol{x}_i) = \pi_{i1} u_i(x_{i1}) + \pi_{i2} u_i(x_{i2}) > U_i(\boldsymbol{x}_i') = \pi_{i1} u_i(x_{i1}') + \pi_{i2} u_i(x_{i2}'),$$

where 'π_{is}' denotes i's (subjective) probability that state s will occur, for $s = 1, 2$. Supposing that these probabilities are strictly positive, and that prices for the two goods at $t = 0$ are given by $\boldsymbol{p} = (p_1, p_2) \in \mathbb{R}_{++}^2$, the i^{th} consumer will maximize utility by setting:

$$\frac{\pi_{i1} u_i'(x_{i1})}{p_1} = \frac{\pi_{i2} u_i'(x_{i2})}{p_2}, \tag{12.11}$$

and:

$$\boldsymbol{p} \cdot \boldsymbol{x}_i = \boldsymbol{p} \cdot \boldsymbol{\omega}_i. \tag{12.12}$$

Assuming that the two consumers agree on the probabilities of the two states (so that $\pi_{1s} = \pi_{2s} \equiv \pi_s$, for $s = 1, 2$, it is easily seen that in competitive (Arrow-Debreu) equilibrium:

$$\frac{u_1'(x_{11})}{u_1'(x_{12})} = \frac{\pi_2 p_1}{\pi_1 p_2} = \frac{u_2'(x_{21})}{u_2'(x_{22})}; \tag{12.13}$$

and thus it is easy to see that the allocation will be Pareto efficient.

Now suppose that:

$$\omega_{11} + \omega_{21} = \omega_{12} + \omega_{22}; \tag{12.14}$$

that is, that the total endowment in the two states is exactly the same. Suppose further that $\pi_1 = \pi_2$; that is, that the both consumers consider the two states to be equally probable. Then (12.13) becomes:

$$\frac{u_1'(x_{11})}{u_1'(x_{12})} = \frac{p_1}{p_2} = \frac{u_2'(x_{21})}{u_2'(x_{22})}. \tag{12.15}$$

Suppose then, by way of obtaining a contradiction, that:

$$x_{11} > x_{12} \tag{12.16}$$

Then by the assumed properties of the u_i functions:

$$\frac{u_1'(x_{11})}{u_1'(x_{12})} < 1.$$

However, it then follows from (12.15) that $u_2'(x_{21})/u_2'(x_{22}) < 1$ also; in which case it follows from the assumed properties of the u_i that:

$$x_{21} > x_{22}$$

as well. But then it follows that:

$$x_{11} + x_{21} > x_{12} + x_{22};$$

which, given (12.14), contradicts the assumption that $(\langle \boldsymbol{x}_{is} \rangle, \boldsymbol{p})$ is a competitive (Arrow-Debreu) equilibrium. A symmetric argument shows that we cannot have $x_{i1} < x_{12}$ for either $i = 1$ or $i = 2$. Therefore, we must have:

$$x_{i1} = x_{i2} \quad \text{for } i = 1, 2;$$

and from (12.15) we see that this implies that we must have $p_1 = p_2$.

Maintaining the assumption that (12.14) holds, arguments similar to those of the above paragraph establish that if both individuals believe the first state to be more probable than the second, then we must have $p_1 > p_2$ in equilibrium.

Next suppose that we have:

$$\omega_{12} = \omega_{21} = 0,$$

but that (12.14) continues to hold (so that we have **private risk**, but we do not have **social risk**). Then it follows from the reasoning above that both consumers fully insure; that is, they each sell off rights to half of their endowments in order to equalize expected consumption in the two states.

Finally, suppose that we have **social risk**; that is, suppose we have:

$$\omega_1 \stackrel{\text{def}}{=} \omega_{11} + \omega_{21} \neq \omega_2 \stackrel{\text{def}}{=} \omega_{12} + \omega_{22}, \tag{12.17}$$

but that $\pi_1 = \pi_2$. I will leave it as an exercise to show that in this situation we must have:

$$(p_1 - p_2)(\omega_1 - \omega_2) < 0. \quad \square \tag{12.18}$$

The scenario involved in the usual interpretation of the model we have been discussing is that all markets operate and are cleared in the initial period ($t = 0$), while all consumption takes place at $t = 1$. There are a couple of points which should be made with respect to this interpretation. First of all, there is a question about *ex ante* versus *ex post* efficiency. Suppose we have an Arrow-Debreu equilibrium, $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$, but that markets are re-opened at $t = 1$, after the uncertainty is resolved, but before consumption takes place.[3] What would happen then? Strictly

[3]The markets in question here are called **spot markets**, while the markets at $t = 0$ are called **forward markets**.

speaking, we cannot say without assuming that preferences are weakly separable on X_{is} and that all consumers' *ex post* preferences are the same as their *ex ante* preferences over X_{is}. However, both of these assumptions seem to be eminently reasonable, and if both are true, then there would be no incentive for trades to take place in this situation. Why is this? Well, each consumer must be maximizing satisfaction, given the expenditure $\boldsymbol{p}_s^* \cdot \boldsymbol{x}_{is}$ at $\boldsymbol{x}_{is}^*$; for if, for some consumer i there were some $\boldsymbol{x}_{is}'$ such that:

$$\boldsymbol{p}_s^* \cdot \boldsymbol{x}_{is}' \leq \boldsymbol{p}_s^* \cdot \boldsymbol{x}_{is} \text{ and } \boldsymbol{x}_{is}' \succ_{is} \boldsymbol{x}_{is}^*,$$

the consumer would have preferred to replace the bundle $\boldsymbol{x}_{is}^*$ with $\boldsymbol{x}_{is}'$ at $t = 0$.[4] Since the allocation in state s is therefore a competitive equilibrium, given the price vector $\boldsymbol{p}_s^*$, it follows that it is also Pareto efficient. Consequently, there are no mutually beneficial trades which consumers can make among themselves after the resolution of uncertainty.

A serious objection to the interpretation of the model which we set out in the preceding paragraph is that it is clearly unrealistic to expect the existence of forward markets in each commodity. However, suppose we have an Arrow-Debreu equilibrium, $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$. If prices in each state are correctly anticipated by all agents, and we have a futures market for only one commodity, with trading only in that one commodity at $t = 0$, then we can achieve that same consumption allocation, $\langle \boldsymbol{x}_i^* \rangle$, if re-trading is possible (at the anticipated prices) at $t = 1$. This remarkable fact was first noted by Arrow [1953]. The formal extension of this idea which we will be studying in the next section was, however, developed by Professor Roy Radner [1968, 1982].

12.3 Radner Equilibrium

For the sake of simplicity, in the remainder of this chapter we will confine our discussion to the context of a pure exchange economy, and we will retain the notation and basic assumptions regarding consumers which were introduced in the previous section; so that consumer i has a preference relation $\succsim_i$ on X_i, and has the initial endowment $\boldsymbol{\omega}_i$, as before. Once again we will deal with a two-period model; with uncertainty being resolved in the second period ($t = 1$). This time, however, we will allow no commodity trading in the first period ($t = 0$). We will, however, introduce the idea of some tradeable assets, which can be purchased (or sold short) in the first period. There are three pivotal assumptions which we will make in this context. First, we will suppose that at $t = 0$ consumers have *expectations* of the prices which will occur (and at which trading will take place) at $t = 1$, for each possible state of nature (each $s \in S$). Secondly, we suppose that all consumers expect the same vector of prices to prevail at $t = 1$ if state $s \in S$ occurs; we denote this vector by '$\boldsymbol{p}_s$,' and we denote the full vector of such prices by '$\boldsymbol{p}$;' that is:

$$\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s, \ldots, \boldsymbol{p}_S).$$

[4] Notice, however, that both weak separability and the identity of ex post and ex ante preferences are needed to make this argument correct.