

# INTRODUCTION TO EQUILIBRIUM ANALYSIS

Variations on themes by Edgeworth and Walras

---

W. HILDENBRAND

*University of Bonn*

A. P. KIRMAN

*University of Warwick*



1976

NORTH-HOLLAND PUBLISHING COMPANY  
AMSTERDAM · OXFORD

AMERICAN ELSEVIER PUBLISHING CO., INC.  
NEW YORK

## An introductory outline

### 1. Introduction

In this chapter we will present in a very simple geometric framework some of the problems that are discussed in the body of the book. Our aim is not to justify our choice of topics, this we will do later, but rather to give an insight into the nature of these topics and their treatment. Reading this chapter may well make the remainder of the book redundant for some readers, those who are bright enough to formalize and generalize the geometric presentation themselves, and those for whom it is sufficient to know what the book is about. However, we hope that there are readers who do not fall into these classes! There may indeed be those who have learned their basic economics so well that they will be thoroughly familiar with the material in this chapter and will want to go straight into ch. 2.

The economic activity that we will analyse throughout the book is that of the exchange of commodities. We will be concerned with a group of individuals each possessing goods who meet to exchange them with a view to their eventual consumption. Each actor in this play has views about the various bundles of goods that he might obtain as a result of the exchange. Our aim is to analyse the possible outcomes of the process and its relationship to the individual's preferences. To give a little more life to these ideas let us start by introducing our basic example.

### 2. Basic example

Consider two individuals, Ada and Bill, whom we shall sometimes refer to, somewhat impersonally, as  $a$  and  $b$ , each of whom possess quantities of two goods, bread and wine. Between them they have three loaves of bread

and three bottles of wine. Now suppose that we wish to represent this situation geometrically. Look at fig. 1.1 where we have drawn a box<sup>1</sup>. The vertical axis represents quantities of wine and the horizontal, quantities of bread. Now, if we measure quantities for Ada from the origin, then any point in the box tells us not only the quantities that Ada receives but also those that go to Bill. This is obvious since we know the total of both goods in the economy. Hence, Bill simply gets those amounts not included in Ada's bundle. No point outside the box would represent a feasible distribution since it would represent a negative allocation of some good to one of

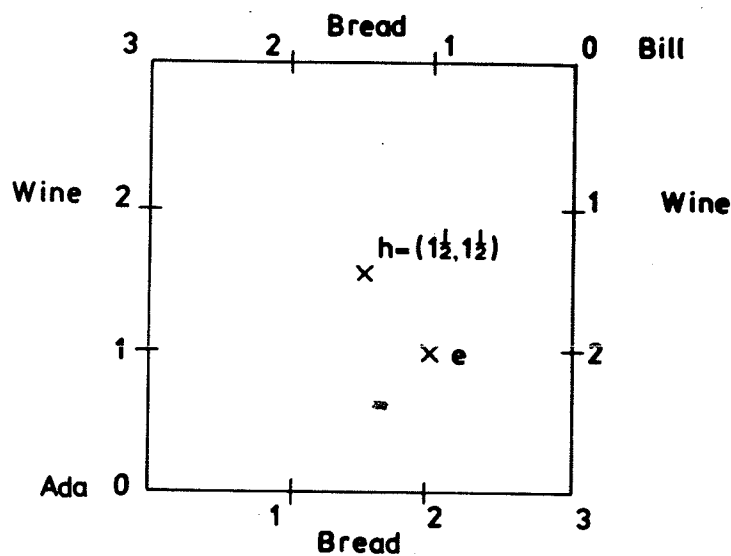


Fig. 1.1.

the agents. To say that every point in the box represents a distribution of the goods implies that we are assuming the goods may be divided up as finely as we wish, they are not only available in 'units'. The size of the box again is given by the total quantities of the two goods that Ada and Bill possess. In our case we have three units of each good, but the box may, of course, be any size. Indeed, if there were only one good it would be a line segment!

We have not yet specified how the goods are originally divided between the two individuals. Suppose, however, that Ada starts with two loaves and one bottle whilst Bill's initial endowments are one loaf and two bottles.

<sup>1</sup>The box device is due to a development by Bowley of an idea of Edgeworth and should properly be referred to as an Edgeworth-Bowley box. However, we will accept current practice with apologies to Bowley's memory and refer to it as an Edgeworth box.

This is the position indicated by  $e$  in fig. 1.1. We could write this a little more formally as

$$e(a) = (2, 1), \quad e(b) = (1, 2).$$

As yet we have said nothing about the outcome of the exchange and if, for example, Ada is very aggressive the result could be point  $f$ , in which she gets everything. This allocation  $f$  we would describe by

$$f(a) = (3, 3), \quad f(b) = (0, 0).$$

Alternatively, Bill might be very persuasive and the final position might be  $g(a) = (0,0)$  in which all the goods go to him. There is, of course, always the possibility that neither would dominate to this extent and the result could, for example, be an equal division of the goods as at point  $h$ . Before we can proceed further we must know more about the desires of the two individuals. Our first assumption is that they are able to make choices between any of the various bundles of goods that they might obtain in the exchange. In other words, consider any two combinations of goods, for example one loaf and two bottles of wine (bundle  $x$ ) and two loaves and one bottle of wine (bundle  $y$ ). Ada and Bill can each make one of the following statements: 'I prefer  $x$  to  $y$ ', 'I prefer  $y$  to  $x$ ', or 'I am indifferent between  $x$  and  $y$ '. These statements may be written symbolically as  $x \succ y$ ,  $y \succ x$  and  $x \sim y$ . In general, we will work with the basic comparison between bundles 'preferred or indifferent to' written  $\succsim$ . We will index this by  $a$  or  $b$  to indicate the individual in question, e.g.  $x \succsim_a y$ .

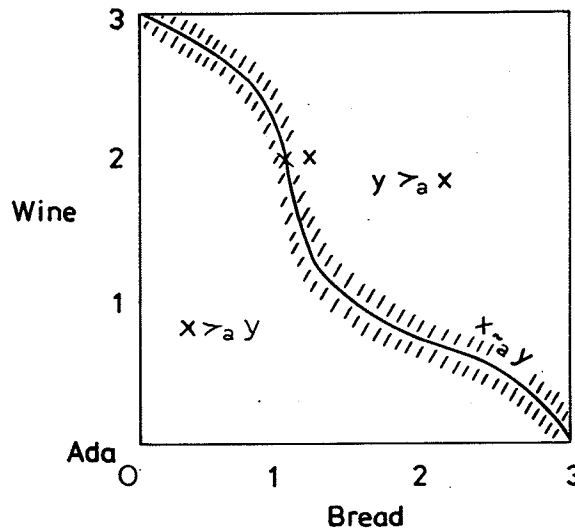


Fig. 1.2.

Now take any bundle for Ada, i.e. some point  $x$  in our box as in fig. 1.2 and consider all the other bundles which for her are equivalent to  $x$ , that is all the bundles  $y$  such that  $y \sim x$  for Ada. Draw a line linking these bundles together and we have the familiar indifference curve of economics textbooks. Assuming that Ada prefers more goods to less, then the set of bundles preferred to  $x$  lies above the curve, to the north-east and those inferior, from her point of view, below and to the south-west. Note that both Ada and Bill are perfectly capable of comparing bundles outside the box, even though these are not available to them. Thus, in fig. 1.3 the

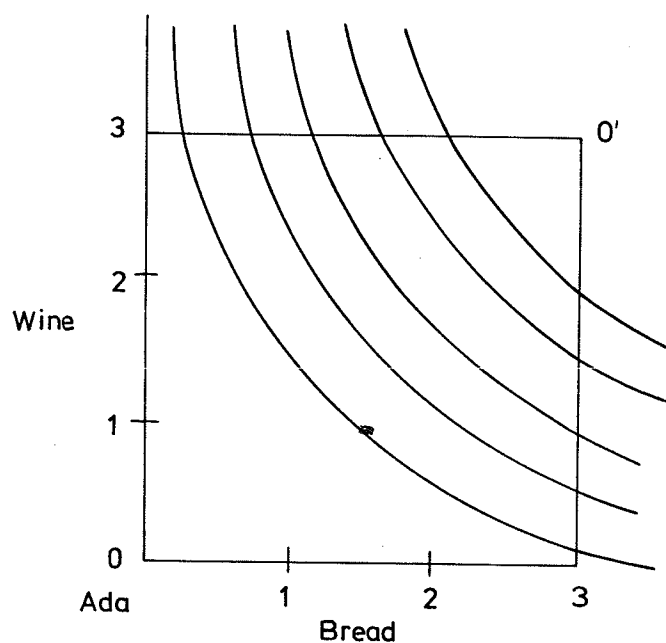


Fig. 1.3.

indifference curves extend beyond the box. However, often we will not bother to extend the curves outside the limits of the box since those redistributions are not possible in our example.

The fact that we can draw nice continuous curves to represent Ada's preferences is, of course, itself an assumption and we will discuss this in some detail in ch. 2. We can now fill the whole box with Ada's indifference curves since every bundle must lie on one of these curves. Bill is equally comprehensive in his ability to choose and his preferences may be similarly represented, observing that the distribution in the north-east corner of the box will be least preferred and the most preferred distributions lie to the south-west from his point of view.

Now observe that we have assumed that goods are desirable, that is that bundles containing more of the goods are preferable to those containing

less, but we have attributed no particular shape to the indifference curves or surfaces in the general case. We will in much of the book add an assumption in this direction. We will assume that these curves are strongly convex. In other words, they have the shape shown in fig. 1.3 for Ada. The basic idea involved in this assumption is that people prefer mixtures to extreme bundles. Thus, if an individual is indifferent between two bundles, he would strictly prefer the average of these two bundles to either of them. The reader will have no difficulty in finding cases where this assumption does not make sense, but we will show that it is not essential to many basic results. However, returning to our example and to the special preferences that we have now assumed for Bill and Ada let us first look at the results of trading.

### 3. First theme: improving by cooperation (Edgeworth)

Our first theme, due to Edgeworth, involves the idea that individuals independently or cooperatively may be able to improve upon redistributions of goods. This simple idea leads to certain redistributions being acceptable as solutions to the exchange problems. Look at our example (section 2) to see how this works.

Let us start with Ada owning all the wine and Bill all the bread, in which case we have  $e(a) = (0, 3)$ ,  $e(b) = (3, 0)$ . Observe first that it is safe to assume that neither will trade if they would finish up with a worse bundle from their own point of view than their initial endowments. Now, looking at fig. 1.4 we see that this must mean that the individuals trade into the

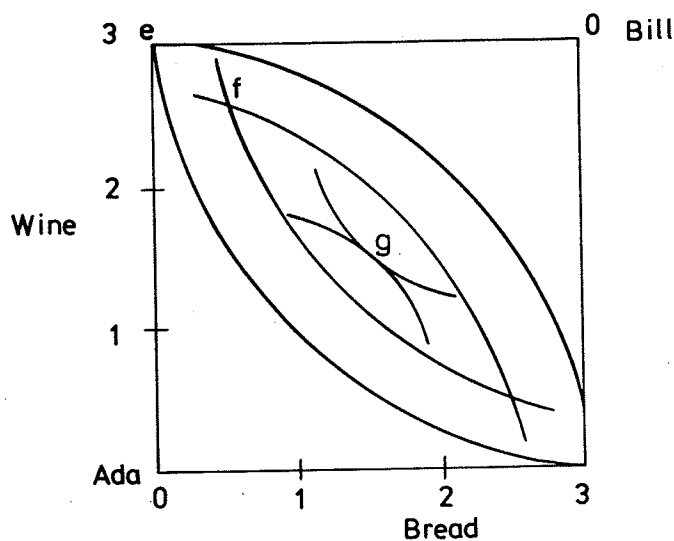


Fig. 1.4.

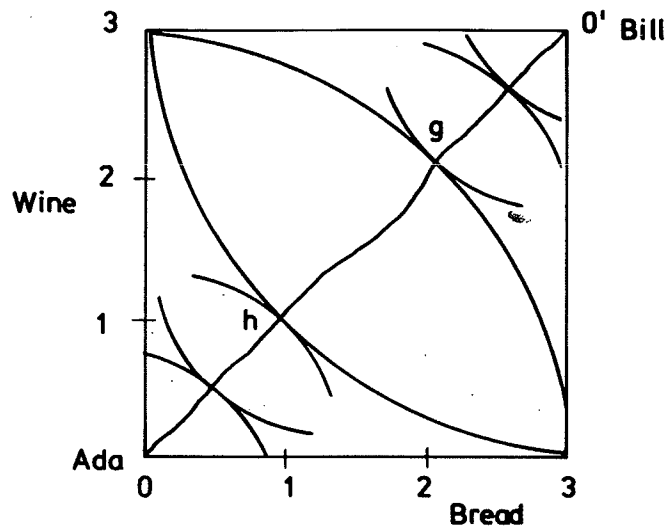
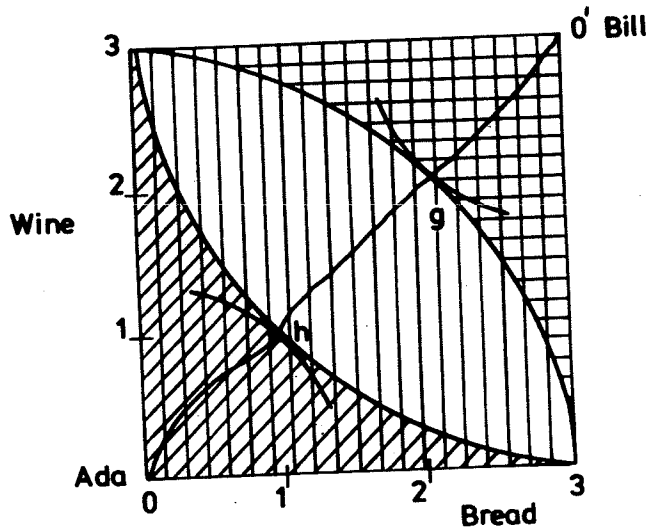


Fig. 1.5.

lens-shaped area in the centre of the box. This area, of course, is the intersection of the set of bundles preferred by Ada to her initial endowments and the analogous set for Bill. As long as this intersection is not empty both Ada and Bill can profit from trade. Suppose that they trade from their initial endowments to a position  $f$  in their preferred set as in fig. 1.4. From  $f$  they may again be able to make mutually profitable trades. However, if they arrive at  $g$  no further trading would be possible without either Ada or Bill being worse off. There are many such positions and the set of all such allocations is known as the set of Pareto optima, which is shown by the line  $0, 0'$  in fig. 1.5. Note that some of these allocations will not be countenanced by Ada and Bill since they place these individuals in a worse position than they were with their initial endowments. Nevertheless, if any of these positions were to occur, the two agents together could not move profitably to any other position.

We have then two types of allocation which are objectionable. Firstly, there are those which can be improved upon by both individuals acting together and secondly, those which either Ada or Bill would refuse since they could improve their lot by simply keeping their initial bundle. The first set of allocations are those off the line  $0, 0'$  in fig. 1.5. The second are those outside the lens in the same figure. The remaining unobjectionable allocations consist of the line  $hg$ , that is the Pareto optima which cannot be objected to by either Ada or Bill. Notice that the set of Pareto optima or 'contract curve', as it is often called, is independent of how the goods are distributed initially. However, the power of Ada or Bill to object is based entirely on what they can manage by themselves. Hence, the lens-






-  Allocations which can be improved upon by Ada
-  Allocations which can be improved upon by Bill
-  Allocations which can be improved upon by both individuals acting jointly

Fig. 1.6.

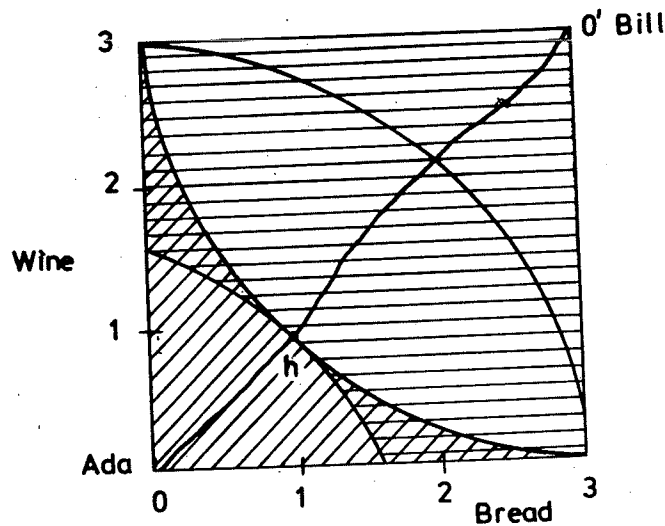


Fig. 1.7.

shaped area depends on the initial endowments, i.e. the initial distribution of 'income', or resources. To see this more clearly, suppose that the initial position had been *h* in figs. 1.6 and 1.7. Then Ada would have objected to any allocation in the diagonally shaded area to the south-west of *h*, but Bill would have objected to anything in the horizontally shaded area to the north-east of *h* in fig. 1.7. Clearly, *h* is the only allocation which could not be improved upon by one of them. Thus, to repeat, the allocations to

which the community as a whole, for the moment Ada and Bill, cannot object, are unaffected by the initial distribution of resources between them. However, the allocations upon which the individuals can improve and hence to which they would raise objections depend essentially upon their initial holdings.

Now with only two individuals in the economy the situation is very simple. There are only two basic sources of objection to proposed allocations: either one of the two individuals or the two acting together. Suppose, however, that there were more people in the economy. How then can we define those allocations which are unexceptionable? The natural extension to our previous discussion is to say that any coalition of individuals can object to a suggested allocation if that coalition could redistribute its initial resources among its members so as to improve the lot of each of its members. In other words, the coalition could *improve upon* the suggested allocation. Now if we eliminate all allocations which could be improved upon in this way we are left with a set of allocations which provide a solution to the problem of exchange. We can summarize this in the following which is our first theme, that attributed to Edgeworth. Its title and the nationality of its author suggest that it would be best played on the *cor anglais*.

*Definition 1.1.* The core of an exchange economy is that set of allocations which can be improved upon by no coalition of individuals.

The core is then a 'solution' to the problem in the sense that it consists of a set of allocations which meet a criterion for acceptability. Whether this criterion is the appropriate one is for the reader to decide. Before proceeding to our second theme, we devote a chapter of the book to showing that, with the sort of assumptions we have already made, there are always allocations in the core. The value of a solution concept is reduced if we cannot specify assumptions under which there are allocations providing such a solution. Once we have specified such assumptions we have to move on to discuss whether they are reasonable and this will occupy our attention later in the book. An assumption which is crucial to the demonstration of the existence of core allocations is that of the convexity of preferences. The reader will be able to think of examples where mixtures of bundles, between which he is indifferent, are indeed worse than those bundles in his eyes, thus contradicting the assumption of convexity. We will discuss this again at length, but for the moment let us simply note the basic role that convexity plays. We will also discuss in ch. 3 what happens to the idea of the core when other difficulties present themselves. If goods are only available in

discrete quantities and cannot be arbitrarily finely divided, for example, then our example looks very different. Suppose that wine can be traded only in units of one bottle and bread in units of one loaf. The possible allocations then look, in our example, as in fig. 1.8. When such indivisibilities are present the core may not exist (and we will give an example in ch. 3). We will, however, also discuss an example where there are core allocations which do have properties of special interest.

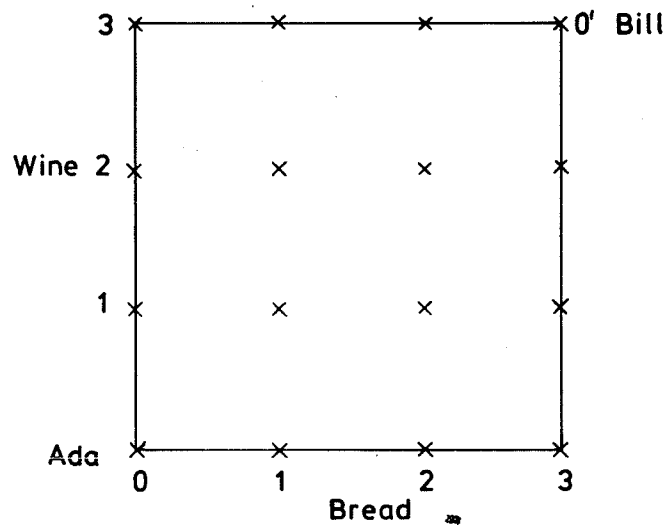


Fig. 1.8.

#### 4. Second theme: decentralization by prices (Walras)

Our second theme, due to Walras, is the decentralization by prices of the exchange problem. The idea, which will be familiar to the reader, is that the individuals in the economy take prices as given. They evaluate what they have at those prices and attempt to obtain the best bundle available which costs no more than their initial endowments. Thus, an individual need only know the going prices and need not cooperate with other individuals. We make the point at the outset that this price-taking behaviour only makes sense when individuals view themselves as an insignificant part of a large market. Thus, our second theme is like Wagner – it can only be played with any success by large orchestras.

One of our major concerns in this book will be to show the close relationship between our two, apparently quite different, themes, and their associated solutions, the core and the *competitive equilibrium*. Before describing this link we have to explain more precisely what is meant by the ‘competitive’

solution. To do this we must change the scenario and the roles of our players. Now, assume that in some mysterious way Ada and Bill know at what prices wine and bread may be exchanged. They both accept the prices as given and only contemplate trades at these prices. Recall our warning that this sort of behaviour cannot make real sense in a two-man world. Indeed, we will show later why this is true. We are doing what is often done in texts interpreting a theme in an inappropriate context. Throughout this discussion, the reader should bear in mind that we are trying simply to explain the nature of a competitive equilibrium, and he should set aside, without forgetting, the problem of inappropriateness.

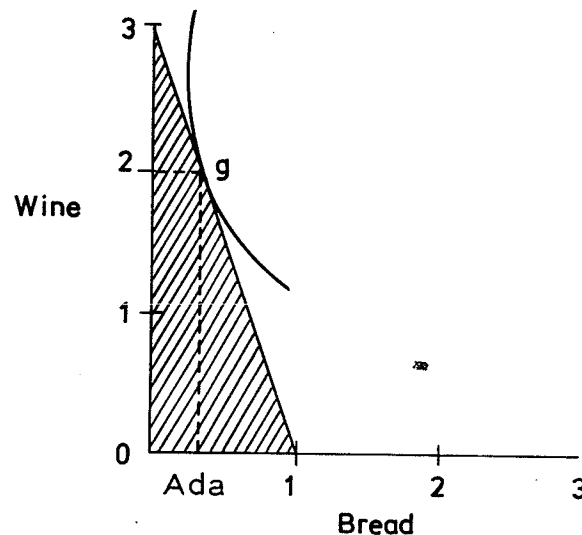


Fig. 1.9.

In the simple two-commodity world of our example we need only one price. We need simply to know the rate at which bread exchanges for wine. Taking the basic price unit in terms of bottles of wine then obviously the price of a bottle of wine is one. In other words, we are setting the price of our second good, wine, as one, written  $p_2 = 1$ . Our interest will be focused on the price of bread, i.e.  $p_1$ . Now let us see how Ada and Bill behave when faced with different prices. Consider the case when  $p_1 = 3$ , that is the price of a loaf of bread is three bottles of wine. Ada as the wine owner can at this price trade all her endowments for one loaf of bread, or make some intermediate trade. Thus, since she may, if she chooses, throw things away, she can consume any bundle in the diagonally shaded area of fig. 1.9. This we can call Ada's *budget set*. Bill, on the other hand, could trade off his three loaves for nine bottles of wine if this much wine were available. His possibilities are then given in fig. 1.10. However, since only three bottles

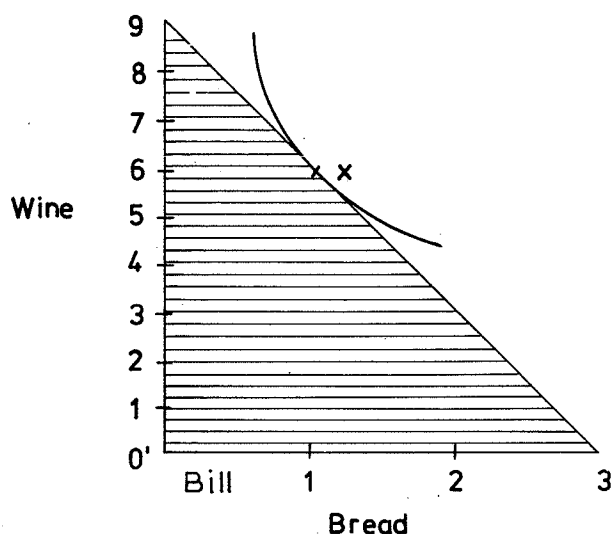


Fig. 1.10.

exist in our economy he could at most trade off one of his loaves for all Ada's wine. No equilibrium could involve a bigger trade.

Now, given the possible trades which will Ada and Bill choose? The answer is, obviously, that which they each find most desirable. In fig. 1.10 we observe that the bundle  $x = (1, 6)$  is the best available for Bill. It is the bundle in his budget set which is on the highest 'indifference curve'. Thus, when the price of bread is three he would choose the bundle  $x$  which is the best that he can afford.

Note that, i.e.

$$p \cdot x = p_1 x_1 + p_2 x_2 = p_1 e_1 + p_2 e_2 = p \cdot e(b)$$

with our values,  $(3, 1) \cdot (1, 6) = (3, 1) \cdot (3, 0)$ . Bear in mind the obvious consequences of our remarks that anything better than what he chose at these prices must cost more. The bundle that he did choose we refer to as his *demand* at prices  $p$ , written  $\varphi(b, p)$  and the difference  $z(b, p) = \varphi(b, p) - e(b)$ , his 'excess demand'. The excess demand is, of course, the trade which he wishes to make at the given prices. Thus, in our example when  $p_1 = 3$  Bill wishes to trade in two of his loaves of bread for six bottles of wine, so given the total supply of wine in our little economy his chances look poor. Ada, on the other hand, faced with the high price of bread, wants to trade to  $g$ , her demand in fig. 1.9. She wants to consume two bottles of wine and one-third of a loaf of bread. Thus,

$$\varphi(a, p) = \left(\frac{1}{3}, 2\right)$$

and

$$z(a, p) = \varphi(a, p) - e(a) = \left(\frac{1}{3}, 2\right) - (0, 3) = \left(\frac{1}{3}, -1\right).$$

Now it is clear that these two trades are incompatible. If we add the trades we see that

$$z(a, p) + z(b, p) = \left(\frac{1}{3}, -1\right) + (-2, 6) = \left(-\frac{5}{3}, 5\right).$$

Thus, there is an excess demand of five bottles of wine and an excess supply of five-thirds of a loaf of bread. If we want the trades to balance, i.e. the markets to clear, we must change the prices. Intuitively, it is apparent that the problem is caused by the wine being relatively too cheap. Now suppose that we can find prices which generate a zero excess demand for both commodities. For convenience assume that Ada's and Bill's preferences are such that when  $p_1 = 1$  they both demand  $(1\frac{1}{2}, 1\frac{1}{2})$ . In other words,

$$\varphi(a, p) = \varphi(b, p) = \left(1\frac{1}{2}, 1\frac{1}{2}\right).$$

Then

$$z(a, p) = \left(1\frac{1}{2}, -1\frac{1}{2}\right) \quad \text{and} \quad z(b, p) = \left(-1\frac{1}{2}, 1\frac{1}{2}\right)$$

and we have

$$z(a, p) + z(b, p) = (0, 0).$$

What precisely does an equilibrium allocation look like in terms of our Edgeworth box? Drawing Ada's and Bill's preferences and budget constraints in the box we have the situation shown in fig. 1.11.

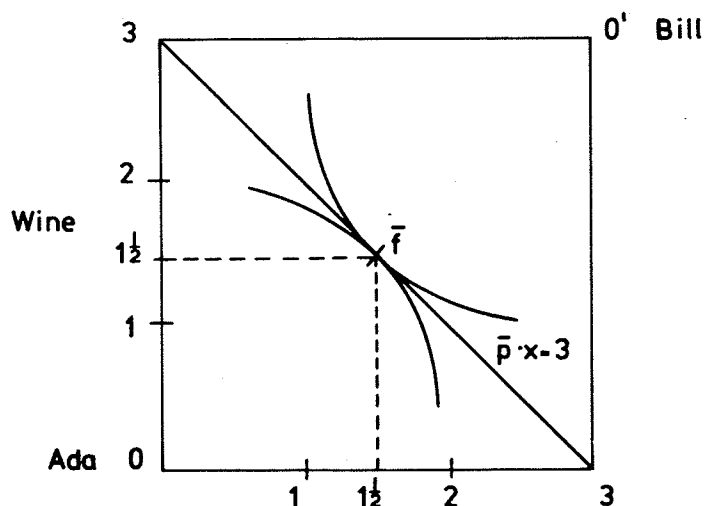


Fig. 1.11.

The indifference curves of the two individuals are tangent at the point  $\bar{f}$  which measured from each origin is the best point in the respective individual's budget set. Recall that Bill's budget set is the area to the north-east of the diagonal price line. The fact that these two best points coincide means that the two agents demand between them exactly the total quantity of goods available. In other words, there is zero total excess demand. Note the price line through the initial endowment point separates the set of allocations which Ada prefers to  $\bar{f}$  and the set which Bill prefers to  $\bar{f}$ .

The first thing to observe about the competitive equilibrium is that it is a Pareto optimum. Furthermore, neither Ada nor Bill would ever choose, at any prices, any allocation worse than their initial endowments. Hence, we know that  $\bar{f}$  could not be improved upon by either of them acting alone and  $\bar{f}$  must be in the core.

Now, at the risk of belabouring the point, recall that, when explaining the competitive equilibrium, we asserted that Ada and Bill behaved as if prices were given to them, and they simply had to accept them. Now, as we have said, this obviously makes no sense for two people since each of them knows that he or she can affect the outcome of the trade and hence can affect the terms on which goods are exchanged.

*Example 1.1.* This example shows how an individual can influence the competitive outcome in a small economy. In this case suppose that Ada starts with four loaves of bread and Bill with three bottles of wine. The competitive equilibrium is shown in fig. 1.12(a) as  $\bar{f}$  where Ada receives

$$\bar{f}(a) = (1, 1\frac{1}{2}) \quad \text{and} \quad p_1 = \frac{1}{2}.$$

Now suppose that Ada throws away a loaf of bread. The new situation is shown in fig. 1.12(b). Now the competitive equilibrium is at  $g$  where Ada receives

$$g(a) = (1\frac{1}{2}, 1\frac{1}{2}) \quad \text{and} \quad p_1 = 1.$$

Clearly, Ada by her apparently bizarre course of action, has influenced the situation in her favour. The reader will recall examples of such behaviour. Coffee was burnt as fuel in trains in Brazil to keep up its price. But the real point is that in small economies the individual's behaviour does affect the outcome.

The only situation where accepting prices as beyond one's influence seems reasonable is in a large market. If there are many participants it becomes less plausible that one individual can have a significant impact

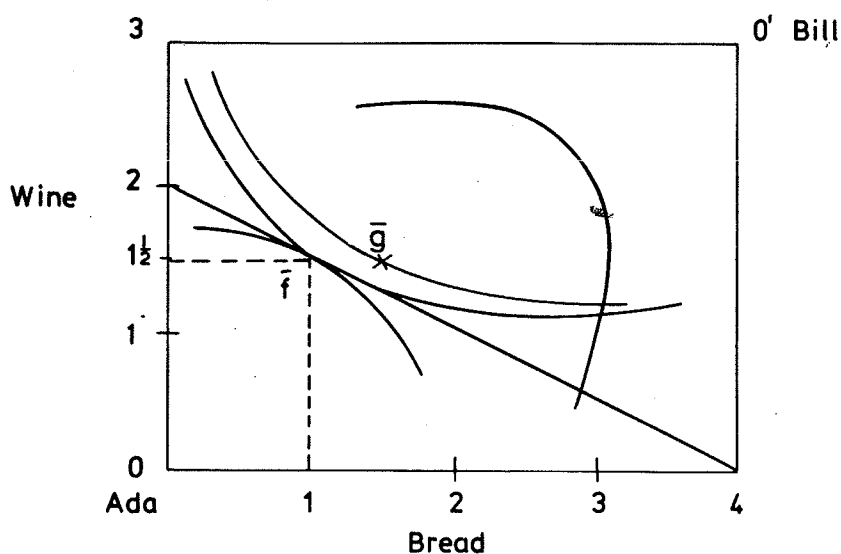


Fig. 1.12(a).

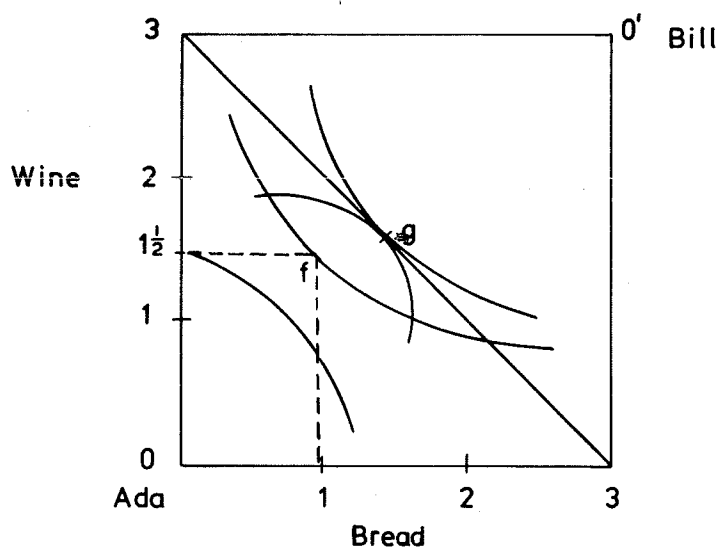


Fig. 1.12(b).

on the market price of a commodity. Of course, if an individual is rich enough he can have such an impact, but this we will discuss in a later chapter. Suppose, for the moment, that we look at an exchange market with many traders none of whom is exceptionally well endowed. This is a situation in which competitive behaviour, i.e. the acceptance of prices as given, makes sense, but what we would like to know is whether the character of our other solution concept, the core, is changed. In fact, what we will show is that in a certain sense the core 'shrinks' as an economy becomes large, and in particular those allocations which are not competitive disappear from the core. The astute reader will be very suspicious of this claim as he will want

to know whether the statement 'the core shrinks' has any meaning. At first sight, it would seem to make little sense. What after all is an allocation? In a two-person economy it is a list of the quantities of each commodity assigned to each of the two traders. In our two-commodity example an allocation is the two pairs of quantities assigned to Ada and Bill. If we add another agent we need to expand our list and an allocation will now consist of three pairs of numbers. How can we decide whether a set of allocations for the two-man economy is bigger or smaller than a set for the three-man economy?

We will now take a very special sort of large economy and show that the statement that 'the core shrinks' is, in fact, meaningful. In ch. 5, however, we will prove results for rather more general economies. It will become clear that these results correspond to the idea of the core reducing to the set of competitive equilibria.

Let us return to our example and increase the number of participants. We will choose a very special way of doing this which will enable us to make statements about the core in large economies. Suppose that, instead of Ada and Bill being alone in our economy, Ada has a twin sister and Bill has a twin brother. These twins not only have identical preferences to those of their counterparts but have identical endowments of goods. Thus, we now have a four-person economy but we only have two *types* of individual. An economist looking at these people would only be able to say 'that woman is an Ada' or 'that man is a Bill'. He would not be able to distinguish between the two women, for example. Remember that the economist is only concerned with two aspects of people, their preferences and their endowments. Ada's sister may well be considerably more attractive than Ada, but if the two women have the same tastes and resources they will be classified, for our purposes, as 'dismal scientists', as identical.

Remember, before we consider the problem of this new four-man economy, that a competitive equilibrium is in the core. The argument is slightly different from the one employed above and carries over to more general economies. The first thing to observe is that if any bundle  $g(a)$  is preferred by an individual  $a$ , to  $f(a)$  his competitive allocation, then at the competitive prices this bundle  $g(a)$  must cost more than his initial resources. This is obvious since otherwise he would have chosen  $g(a)$  at the competitive prices and  $f(a)$  would not have been the best available. Now, if some coalition  $S$  can improve upon the competitive allocation, they must be able to redistribute their initial resources amongst themselves to find for each individual something, say  $g$ , preferred to  $f$ . But if  $g$  is

preferred to  $f$  by all the individuals in  $S$ , then  $g$  must cost more for each of them than their initial bundles. Hence, in total,  $S$  by redistributing its resources amongst itself achieves something higher in value than those resources – an obvious nonsense! Thus a competitive allocation must be in the core.

Furthermore, note that the competitive allocation must assign identical bundles to the two Adas. Obviously neither can get something better than the other since they both choose from the same budget set and have the same preferences. Neither can they get bundles which are different but lying on the same indifference curve, for with the strongly convex preferences we have assumed the average of the two bundles would be strictly better than their choice and still affordable. Hence, they could not have chosen the best bundle in the first place. The same obviously applies to the two Bills. This takes us on to an important point. If we are interested in competitive allocations in our four-man economy we need look only at the allocation of one of the Adas and that of one of the Bills. In other words, the competitive allocation for this economy can still be described in our familiar Edgeworth box.

The question now arises, is this true for core allocations as well? In other words, if a core allocation gives identical bundles to people of the same type then we just choose one of each type, in our case one Ada and one Bill, and by looking at their allocations we know all there is to be known about the allocation for the whole economy. If this is the case in our four-man economy we can still concentrate our attention on the Edgeworth box and need only consider allocations in that framework. This gives a certain sense to the idea of looking at the core as the number of participants if an economy grows.

The argument that the core does have this *equal treatment* property is most easily given in the context of our example and extends directly to the more general situation discussed in later chapters.

Suppose that there is an allocation  $f$  which does not assign identical bundles to identical individuals, can it be in the core? There are two possible ways in which this unequal treatment could arise. Suppose that Ada ( $a, 1$ ) and her sister ( $a, 2$ ) receive bundles which are different but which they rank equally. In other words,

$$f(a, 1) \neq f(a, 2) \quad \text{but} \quad f(a, 1) \sim_a f(a, 2).$$

Clearly, with strongly convex preferences the average of these two bundles would be strictly preferred by both Ada and her sister, and the allocation  $g$

which gives this to the two females and the previous  $f$  to the two Bills would represent an improvement, is obviously feasible, and therefore provides the basis for an objection to  $f$  by the whole group. Hence,  $f$  is not in the core.

The second possibility is that there is unequal treatment in a stronger sense. Thus, for example, if Ada 1 were to receive something worse than Ada 2, could the allocation  $f$  be in the core? Now one of the two Bills must be no better off than the other: suppose that this is Bill 1. They may well be equally treated in this allocation of course. Bill 1 and Ada 1 can surely achieve for themselves the average Bill and Ada allocation respectively. But this must be strictly better for Ada 1 and can be no worse for Bill 1. Therefore, these two can improve upon  $f$  and  $f$  cannot be in the core.

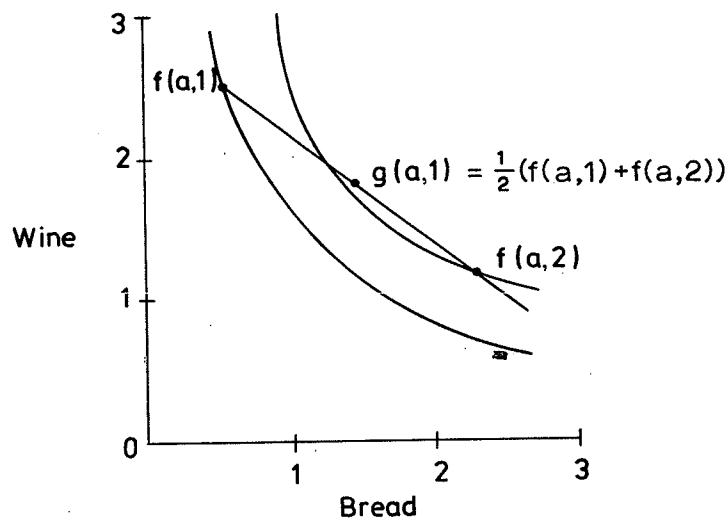


Fig. 1.13.

The fact that the average  $g$  of the two Adas' bundles will be preferred to the worst of these bundles is, of course, a definition of the convexity of preferences and is illustrated in fig. 1.13. We may summarize this argument as follows:

$$\sum_{i=1}^2 f(a, i) + \sum_{i=1}^2 f(b, i) = 2e(a) + 2e(b),$$

$$f(a, 2) \succ_a f(a, 1) \quad \text{and} \quad f(b, 2) \succeq_b f(b, 1).$$

$$\frac{1}{2} \sum_{i=1}^2 f(a, i) \succ_a f(a, 1) \quad \text{and} \quad \frac{1}{2} \sum_{i=1}^2 f(b, i) \succeq_b f(b, 1),$$

but

$$\frac{1}{2} \sum_{i=1}^2 f(a, i) + \frac{1}{2} \sum_{i=1}^2 f(b, i) = e(a) + e(b).$$

Therefore,  $(a, 1)$  and  $(b, 1)$  can improve upon  $f$ . The fact that we only have two Adas and two Bills is obviously immaterial. If we had some arbitrary number  $n$  of both we could still use this argument. Just order the people of each type in terms of the satisfaction derived from their bundles and the worst off of each type will be able to improve upon any allocation with unequal treatment.

All this shows that if we have equal numbers in each type then it is enough, when considering core allocations, to look at the allocation to one representative of each type. Thus, no matter how many identical replicas of Bill and Ada we have, we may continue to represent core allocations in our original Edgeworth box.

*Observation.* To return to our original analogy, what we have said is that if we have a certain number of instruments in the orchestra then for our purposes it is enough to look at the score for one of each instrument. It does not matter how many first violins there are since they are all playing the same thing. Our statement is in fact more restrictive than this in that we are really saying it does not matter if we replicate the orchestra, but it does matter if we increase some instruments and not others.

In our example, had Ada had a sister and Bill no brother then there might well be core allocations which did not treat Ada 1 and Ada 2 equally. We will show an example of this in the text. However, for the moment we shall focus on replications of our Bill-Ada economy.

## 5. The coincidence of two themes

In our four-man economy let us try to see which are the core allocations. We may, for the reasons just given, look at the Edgeworth box (fig. 1.14). Now it is clear that the allocations in the core must be 'contained in' the old core of the two-man economy. Otherwise, whoever objected in the two-man situation would object in the four-man framework. The question is, are there allocations which were in the core originally but now, with the introduction of two new players, disappear?

Look at the allocation which in the original economy was the worst in the core for Ada. It was that which was on the same indifference curve as her initial resources. Could this allocation, now to two Adas and to two Bills, be in the core? The answer is no, and the reasoning is simple. Think of the coalitions that could form in the two-man economy. They were Bill

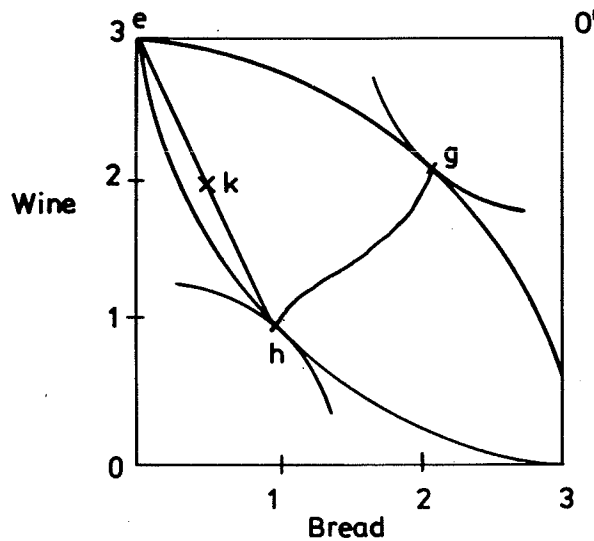


Fig. 1.14.

or Ada alone or the two of them together. Now, however, we have a new possibility. The coalition of two Adas and one Bill will ensure that the old endpoint of the core is now removed. It will make sure that any allocation in the core gives something better to each Ada than her initial resources. Brotherhood or sisterhood is not to be despised!

How does this coalition improve upon the allocation in question? Note first that any point on the line between the initial Ada resources and the point  $h$  would be strictly preferred by Ada to her initial bundle. In particular, this is true of the halfway point  $k$  in fig. 1.14. Thus,

$$k(a, 1) = k(a, 2) = \frac{1}{2}e(a) + \frac{1}{2}h(a, 1) \succ_a h(a, 1) = h(a, 2). \tag{1}$$

But consider the allocation for the three-man coalition (the two Adas with, say, Bill 1) which gives  $k$  to the Adas and  $h$  to Bill 1. This is feasible since

$$k(a, 1) + k(a, 2) + h(b, 1) = e(a) + h(a, 1) + h(b, 1)$$

and  $h$  was an allocation for the two-person economy, i.e.

$$h(a, 1) + h(b, 1) = e(a) + e(b).$$

Therefore,

$$k(a, 1) + k(a, 2) + h(b, 1) = 2e(a) + e(b).$$

Then the fact that Bill 1 still receives  $h(b, 1)$ , together with eq. (1), means that this coalition, two Adas and a Bill, can improve upon  $h$  and that allocation is not in the core of the four-person economy.

In fact, as is clear from fig. 1.15, every allocation in the old core below,

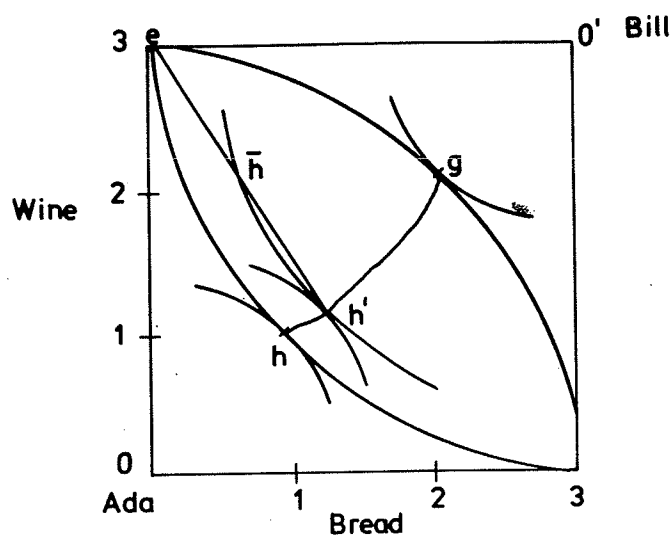


Fig. 1.15.

from Ada's point of view,  $h'$  is ruled out in this way. Now  $\bar{h}$ , which is the average of the initial resources and  $h'$ , lies on the same indifference curve as  $h'$  and can no longer be improved upon by the three-person coalition. A similar argument using a coalition of two Bills and an Ada removes allocations at the other end of the core.

Now we make the argument to show that all allocations in the core except the competitive equilibrium are removed if we replicate our economy still further. Thus, if we add more Adas and more Bills all non-competitive allocations disappear from the core. Concentrating our attention on those allocations below, i.e. to the south-west of the competitive allocation, we notice the following critical fact. The line  $e$  to any non-competitive allocation  $f$  must cut the indifference curve passing through  $f$  and must be above that curve near  $f$ . Thus, there must be a point close to  $f$  on that line which is preferred to  $f$  by Ada. In other words, for any  $f$  if we look at a bundle

$$g(a) = \frac{1}{n} e(a) + \frac{n-1}{n} f(a)$$

this would, if  $n$  were very large, have the property that

$$g(a) \succ_a f(a).$$

We drop the index  $i$  since all people of the same type are equally treated in  $f$ . This is illustrated in fig. 1.16.

But the coalition consisting of  $n$  Adas and  $n-1$  Bills can improve upon  $f$  by giving  $g$  to the Adas and  $f$  to the Bills. We have only to show that this

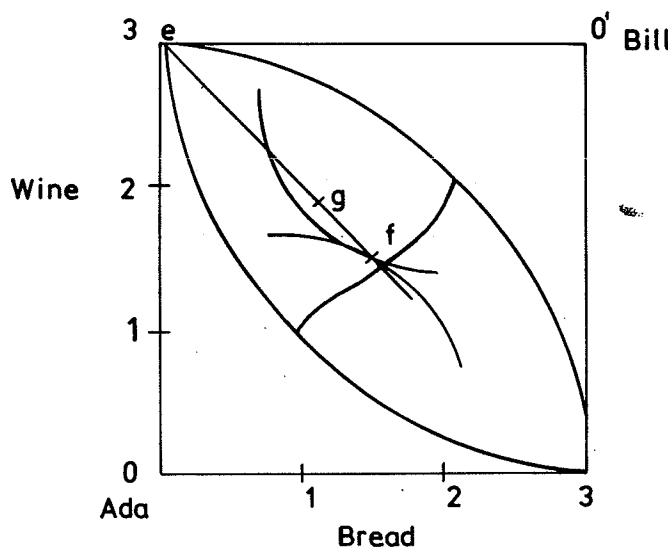


Fig. 1.16.

is feasible for such a coalition. We know that

$$\begin{aligned}
 n[g(a)] + (n-1)[f(b)] &= n \left[ \frac{1}{n} e(a) + \frac{n-1}{n} f(a) \right] + (n-1)[f(b)] \\
 &= e(a) + (n-1)[f(a) + f(b)] \\
 &= e(a) + (n-1)[e(a) + e(b)] \\
 &= n[e(a)] + (n-1)[e(b)].
 \end{aligned}$$

Hence,  $f$  is not in the core. Similarly, on the other side of the competitive allocation the coalition of  $n-1$  Adas and  $n$  Bills will improve upon any allocation that is not competitive. The competitive allocation is just that where the line from the initial resources to the allocation does not cut either of the indifference curves passing through the allocation. It is thus the only allocation which cannot be improved upon in the way described.

Note what we have done. We replicate *ad infinitum* (*ad nauseam* if you will) our Ada and Bill economy. In so doing we know that eventually we will have  $n$  Adas and  $n$  Bills, with  $n$  sufficiently large to remove any non-competitive allocation in the original core. Recall again that we can always look at allocations in the original Edgeworth box since we know that core allocations in these replica economies have the equal treatment property.

The observant reader will have noted something further. The procedure we have just described gives us a way of defining competitive allocations apparently quite different from the definition that we gave earlier.

*Characterization.* An allocation is competitive for a given economy if and only if it remains in the core for every replication of that economy.

There is a little looseness of language involved which the reader will probably find is more than offset by the occasional insistence on precision in later chapters! When we say 'remains in the core' we mean the replication of the allocation, i.e. giving that allocation to every member of each type remains in the core.

The important thing to observe is that this definition in no way involves the idea of prices, yet it is by means of these prices that competitive allocations are normally defined. It is a remarkable fact that the Walrasian idea of a solution to the problem of exchanges comes to coincide with the Edgeworth solution as an economy is expanded in the way that we have described. Put another way, those allocations which persist as acceptable from the Edgeworth viewpoint are precisely those which Walras singled out as solutions.

We have said previously that the competitive equilibrium is, like Wagner, appropriate for large orchestras. Recalling that the number of possible coalitions in economies with many agents is large we might feel that the core is less plausible in this context. Thus, the core is like the music of Corelli, more suitable for smaller orchestras. If the orchestra is large enough, however, Corelli begins to sound like Wagner!

One might be tempted to make the following argument on the basis of our results. The core is a cooperative solution, no group or individual can improve upon its allocations, therefore it is democratic or socially acceptable. The fact that the only allocations which remain socially acceptable turn out to be competitive, adds weight to the idea that the competitive allocation has social merit as a solution. The reader will see the hole in this reasoning very quickly. The objection to the competitive allocation, from a social viewpoint, is that it depends upon the ownership of the initial resources. In other words, 'unto those that hath shall it be given'. Exactly the same objection applies to the core, it is democratic, *given the initial distribution of resources*. The latter may be very unequal and if this is the case, the inequality may persist in the core allocations.

Let us backtrack a little and note what we have shown. We have discussed what happens to the core when economies become large in a very special way, that is are replicated. We have given a geometric argument to show how this happens. This argument, however, is only valid for the two-type, two-good case, but we will give a different argument for the general case in the text. The change in the core with the increase in the size of the economy enabled us to characterize competitive allocations.

Despite our denial of any special social content of this fact we will emphasize yet again its most interesting feature. If one asks under what

circumstances would an individual accept prices as given, the answer is, as we have suggested, standard. He will accept prices and act as a price taker when he feels that he can have no effect on those prices; in other words, if he is an insignificant member of the market. What happens when we replicate an economy? The individuals in that economy become insignificant. Thus, the coincidence of competitive allocations and core allocations occurs under precisely those circumstances when competitive behaviour makes sense.

Rephrasing our result we can say, if we take a given economy and replicate it until all the individuals become insignificant, that the only allocations in the core of this enlarged economy are competitive.

## 6. A conjuring trick

A question that preoccupied Walras and many of those who followed him was under what assumptions does a competitive equilibrium exist? The brief answer is that in the sort of economy illustrated by our Ada and Bill example, a competitive equilibrium, that is a competitive allocation and the associated prices, always exists. We will sketch here an argument to show that this is so, an argument sufficiently different from the standard one that it may seem to old hands at mathematical economics something of a rabbit out of a hat. This is not the case, but showing that our argument boils down to the traditional one may occupy a few minutes.

The basic result of ch. 3 is that, for the sort of economies of which our two-person case is an example, the core is never empty. That is, there is an allocation which can be improved upon by no coalition in such an economy. Now think of our replica economies, the two-man, four-man, six-man, etc., sequence. Associated with each of these economies will be a core allocation  $f_1, f_2, f_3, \dots$ . In each case consider these allocations as to just one of each type, i.e. the allocation to one Ada and one Bill. Remember that this is legitimate since the  $f$ 's are core allocations and members of the same type are equally treated. Now it is easily shown that there is a subsequence of this sequence of allocations which converges to something  $\tilde{f}$ . It is also easily shown that  $\tilde{f}$  is in the core for every  $n$ -fold replica economy. Therefore,  $\tilde{f}$  is a competitive allocation for the first economy.

To see this argument in terms of our original box consider what we were doing. We took some core allocation in the first economy, i.e. some allocation on the segment of the contract curve between the two indifference

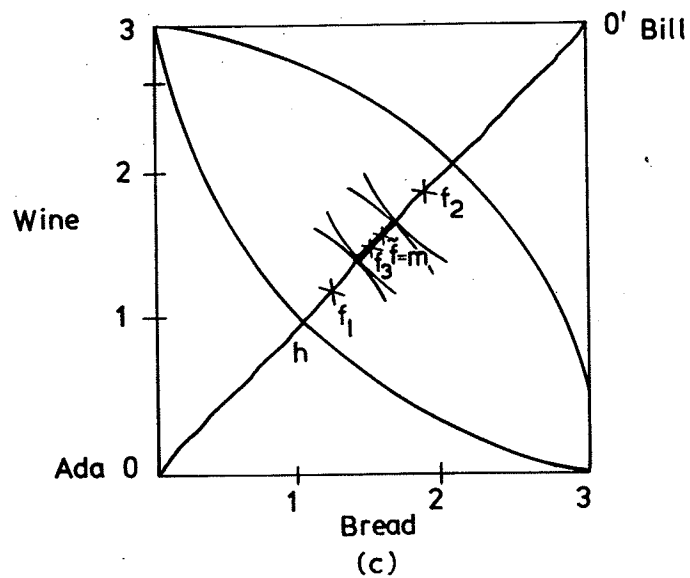
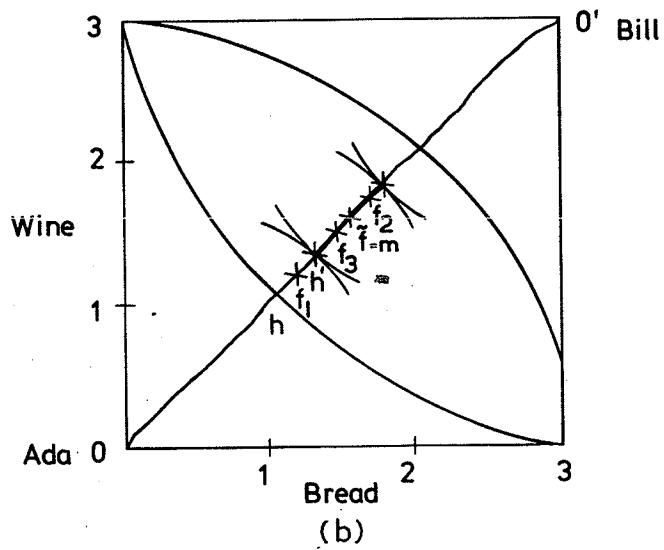
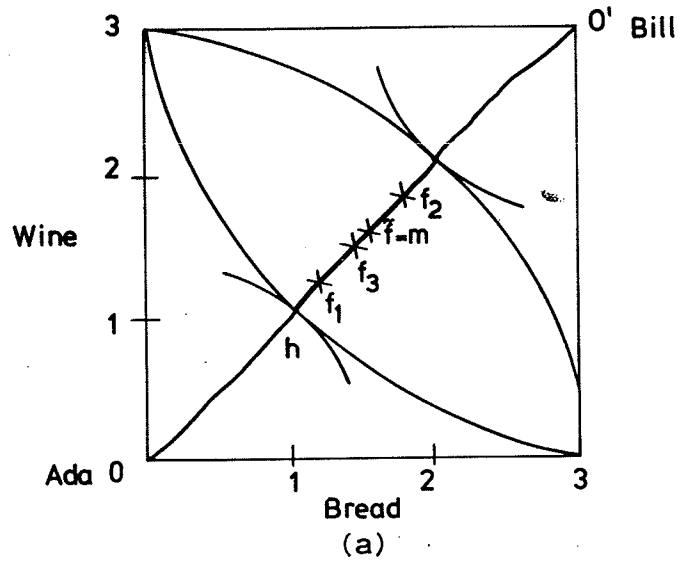


Fig. 1.17: (a) Original core, (b) core of four-man economy, and (c) core of six-man economy.

curves which passed through the initial endowment point. This was  $f_1$ . Then we took an allocation from the core of the four-man economy, that is from the subset of the original core shown in fig. 1.17. Then  $f_3$  is taken from the core of the six-man economy. This sequence of points must converge, of course, and they do, to the competitive allocation, thus showing the existence of such an allocation. Using our previous argument we know that the limit of our sequence must be a point on a line from the initial resources just separating the preferred sets of Ada and Bill on the contract curve, i.e. the line which provides the competitive prices for the allocation.

We will return to the Walrasian theme shortly, but first let us expand the remarks above about insignificance.

### **7. Limit economies and the importance of being insignificant**

We talked about replicating an economy until the individuals in the economy were insignificant. For competitive behaviour (passive behaviour might be a better description) to make sense, individuals must believe themselves to be insignificant. Now, in general, in an economy with a finite number of individuals, the latter will be able to change prices, for example, by altering their own behaviour as we showed in our earlier example. However, we can talk about economies where individuals would not have this capacity. Thus, we can define an economy where each individual has literally no weight or influence, where the removal of an individual would change nothing. Such an economy should not be thought of as a description of a possible economy but rather as an idealization in the same way that physicists might think of perfect fluids, rather than of the individual particles of which these fluids are composed. As an example, we might think of the agents in an economy as corresponding to the points on the unit interval. There is then a continuum of agents each having no 'weight'. For such an idealized economy it can be shown that, under rather weak assumptions about the characteristics of individuals, the core and the competitive allocations are the same. Note, of course, that in this sort of economy as in any other there may be several equilibrium allocations. We will return to this point later. The point is that in these rather rarefied economies we have the equivalence towards which we worked with our replication arguments. The final and perhaps most important remark about these continuum economies is that they may be regarded not just as idealizations but also as limits of a sequence of finite economies. This leads us directly to the next idea.

### 8. From limits to limiting results

Once we can talk about a sequence of economies approaching a limit then we talk about the 'convergence' of the core to the competitive equilibria in a simple way. Suppose that we remain in the Ada and Bill world for a moment. Now instead of imagining this world expanding by a simple replication, could we answer the following question? As the Ada and Bill world becomes big is it true that core allocations are near to competitive allocations? The Ada and Bill world may not become big in the way that we specified previously. More Bills may be added than Adas and so forth. You could imagine a sequence of increasing economies being generated as follows. Take a deck of cards with all the kings and queens removed. Now draw a card. If it is an odd number, an ace counts as one, a jack as eleven, then add that number of Adas to the economy. If it is even, do the same for the Bills. At any moment in this procedure the proportions of the two types will be fairly arbitrary, but if we do it long enough we will have the proportion of Adas converging to 25/55 and that of Bills to 30/55.

Let us take a core allocation for each of these economies and look at the sequence of these. We run into trouble straight away because our 'equal treatment' property has disappeared. People of the same type may receive different bundles in a core allocation if there are not equal numbers in each type. We give a simple example of this in the text. How can we get over this difficulty? What we can do is to use a utility function for Ada's preferences and one for Bill's. A utility function is just a numerical representation of preferences. Bundles which are preferred to a given bundle  $y$  are assigned higher numbers than that given to  $y$ , i.e.  $u(x) \geq u(y)$  if and only if  $x \succeq y$ . Clearly, bundles which are indifferent receive the same number. That it is

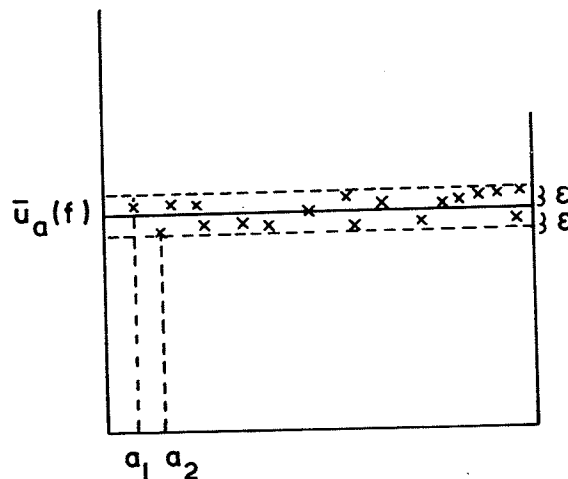


Fig. 1.18.

possible to represent preferences by such a function is shown in ch. 2. There are, of course, many functions which represent given preferences, but in this case we will choose one for Bill,  $u_b$ , and one for Ada,  $u_a$ . Using these functions we can say that for any small epsilon, when the Ada and Bill economy is large enough, no one's utility in a core allocation will differ by more than that epsilon from the average utility  $\bar{u}(f)$ , of his type. Hence, the picture of a core allocation in a large economy for one type must look as in fig. 1.18.

Thus, the utility of a core allocation for a member of one type is converging to the average of that type. Furthermore, with the strongly convex preferences that we have assumed the bundles of each type are approaching the average. Our argument from this point is a simple one. We now look at the average bundles of each type. We are back again in the representative situation, one bundle for each type. This gives us a sequence which converges to a core allocation in the limit economy. Hence, it converges to a competitive allocation in that economy. Now consider the prices associated with this competitive allocation in the limit economy. Since each individual's bundle is approaching that in the limit allocation which is optimal for him at those prices, then in large economies his core allocation must be close to optimal for him at the same prices. Thus, we have the convergence of the core to the competitive equilibria that we required.

To sum up, in our simple example, as the economy becomes large all Adas receive bundles close to the average Ada bundle in that economy. The same applies to the Bills. These average bundles for Ada and Bill converge in turn to something which if given to all Adas and Bills would be a competitive allocation in the limit economy. If the prices associated with this limit economy competitive allocation were called out in a large economy in the sequence all individuals would receive in the core allocation for that economy something very close to the best bundle they could choose at these prices. Thus, core allocations in large economies can be decentralized by prices. The important thing to see here is that our results confirm the impression that the limit or continuum economy does have something important to tell us about large but finite economies. We have continuity, in the sense that what is true in the limit is almost true in very large economies.

### 9. Back to Walras

Up to this point the Walrasian theme, that of decentralization by prices, has only emerged in connection with the core and for many readers this must seem a little bizarre. The mainstream of mathematical economics has focused firmly on the problem of demonstrating the existence of competitive equilibria and it is only very recently that Edgeworth's approach has encroached upon the scene.

Let us then revert to the traditional approach and see how one might set about establishing existence without even a glance over the shoulder at game theory. To do this we will have to concentrate on the demand of the

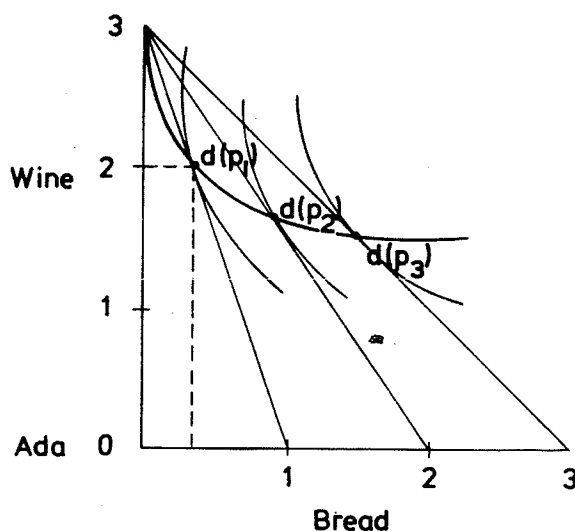


Fig. 1.19.

individuals in the economy. Recall that the choice of an individual at given prices is the bundle that he most prefers within the budget set.

Taking Ada to start with, in our example, we could at all possible relative prices of bread and wine trace out her choices or 'demands'. Look at fig. 1.19 and the curve traced out in this way is illustrated. In our example Ada's behaviour corresponds to intuition in that, as the relative price of bread falls, she consumes more and more of it, giving up some of her wine in exchange. We could now repeat this exercise for Bill and we would have a similar curve. Putting them back together in the Edgeworth box it is easy to see that, from what we have said before, equilibrium occurs where these two curves cross. Thus, in fig. 1.20 we see that the equilibrium prices in the example are given by  $\bar{p} = (1, 1)$  or, since we need only one price,  $\bar{p}_2 = 1$ . The equilibrium allocation corresponding to Ada's and Bill's demand at these prices are  $f(a) = (1\frac{1}{2}, 1\frac{1}{2})$ ,  $f(b) = (1\frac{1}{2}, 1\frac{1}{2})$ .

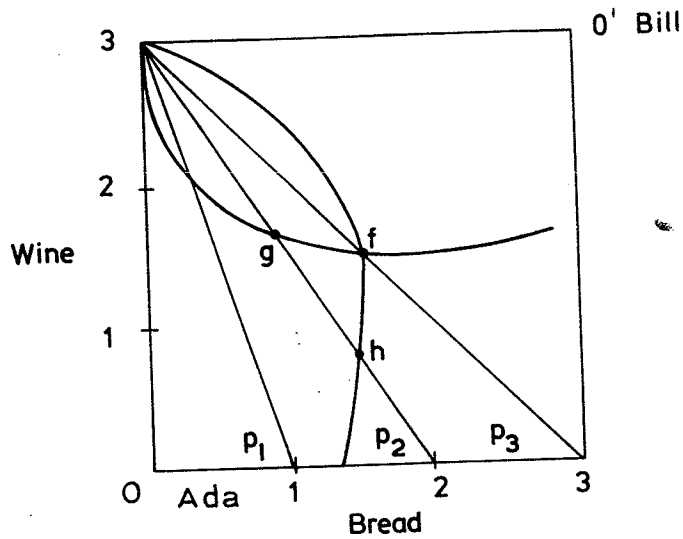


Fig. 1.20.

The point of intersection of the two demand curves, or offer curves, as they are known in international trade theory, is, of course, that which we have already seen. Looking at these curves we can see the situation out of equilibrium. Look at  $p_2$ , Ada wants to consume one and two-thirds units of wine and eight-ninths of a unit of bread,  $g$ , whereas Bill wants to consume two and a quarter units of wine and one and a half units of bread,  $h$ . Obviously, bread is, at a price of one and a half, too expensive. There is excess demand for wine and an excess supply of bread. To remedy this undesirable situation it is enough to lower the price of bread to one and thus to create an equilibrium situation.

To prove existence in an economy such as that in our example is a matter of showing that the two curves do, in fact, cross in the box. It is easy to see that this must be so. Both curves start at the north-west corner of the box. This point does not, however, constitute an equilibrium since the indifference curves of Ada and Bill through this point intersect elsewhere in the box. Thus, the point represents demands at different prices for Ada and Bill respectively in our example. This may readily be seen by constructing the indifference curves corresponding to this point for Ada and Bill. Now Bill's offer curve must touch the bottom of the box. It cannot cut the right-hand edge since then there would be bundles available at some prices which would contain strictly more bread than that chosen by Bill. This would be a violation of the monotonicity or greed assumption. For the same reason Ada's curve must cut the right-hand edge of the box. Now, given that the curves are continuous, they must intersect.

The assiduous reader will immediately construct an example unlike ours

where the only point at which the curves intersect is the north-west corner, i.e. where each has his initial endowment. If this is true then it is easy to show that in such a case the point must be an equilibrium. That is the indifference curve of the two individuals through this point, unlike those of our example, do not intersect elsewhere in the box and they can be separated by a price line through this point.

*Remark.* It is easy to construct examples where there are several equilibria. Indeed it is possible to have an infinite number of equilibria as in fig. 1.21.

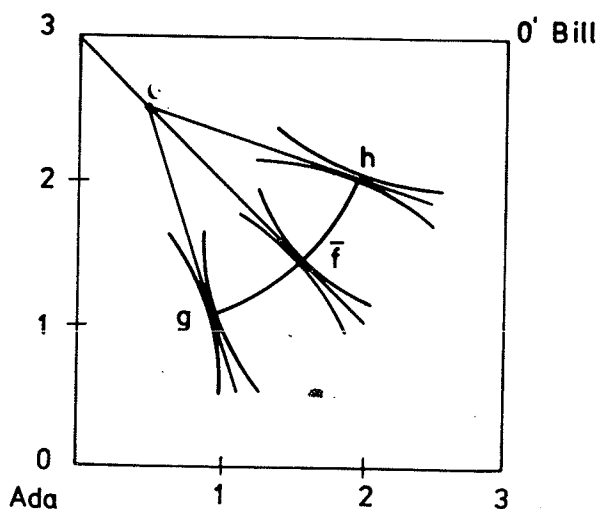


Fig. 1.21.

In this example if Bill and Ada start at  $e$  then every redistribution on the contract curve from  $g$  to  $h$  is a competitive equilibrium. Note, however, that moving  $e$  a little would remove all of these equilibria except at most one. In fact, others might appear, but it is possible to show that for 'almost all' initial endowment positions there are only a finite number of equilibria.

It is evident that if the offer curves had had gaps, for example, then our argument would not have held. These gaps cannot occur if we have convex preferences. If there are no such gaps then demand behaves continuously, as prices change. This continuity of the demand relation is at the heart of the existence proofs in ch. 6. In our example and in the literature in the Walrasian tradition this assumption of convexity of the individual's preferences is necessary. Indeed, the basis of the arguments used in all the standard proofs of existence is that aggregate behaviour has the continuity properties of individual behaviour. Thus, if individuals are 'well behaved' then so will the economy be.

## 10. Less well behaved individuals

Suppose that we had made fewer assumptions about Ada and Bill. In particular, suppose that we had not assumed them to have convex preferences. After all many people might be indifferent between a main course of steak or one of duck, but there are few who would find a mixture of the two attractive. In our example with bread and wine preference for extreme bundles would be difficult to justify, but if there are many goods it is easy to see how such choices might arise. In the simple two-good world bread and wine would be needed for survival, but if we extend the range of goods then there will be many consumers who dislike certain mixtures and prefer to eliminate certain commodities from their consumption.

That this presents a problem is clearly seen by changing our example. Suppose that Ada, instead of being the amiable convex person of earlier pages now becomes more extreme. If her preferences now look as in fig. 1.22 we see that she is indifferent between the two bundles  $(0, 3)$  and  $(3, 0)$ . However, far from preferring intermediate bundles any mixture of the three-loaves bundle and the three-bottles bundle is worse for her. Now as long as wine is cheaper than bread she will always choose to consume all

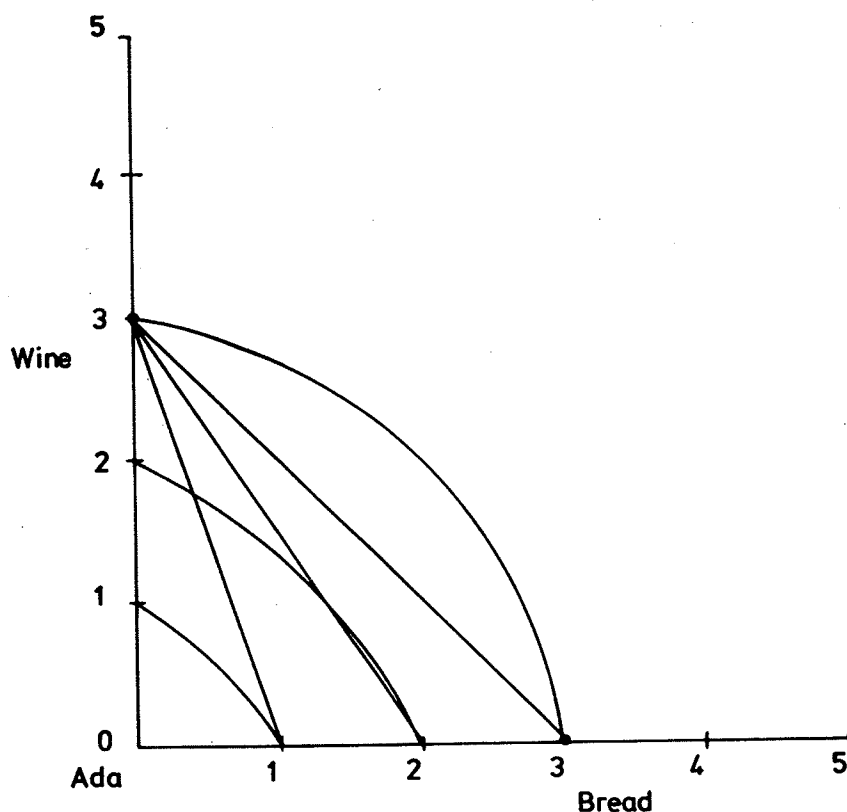


Fig. 1.22.

wine. Yet when the prices of these two commodities are equal she will choose either three bottles of wine or three loaves of bread and as bread becomes cheaper she will want only bread. Her offer curve is no longer a nice continuous curve, but with equal prices jumps from one side to the opposite side of the other curve. There is then no equilibrium in this case. The lack of convexity led to a lack of continuity of Ada's offer curve and thus destroyed our hope of finding an equilibrium. Supposing, however, that we had modified Ada's demand somewhat by 'convexifying' her demand. Thus we would modify her choice at equal prices of wine and bread to include all bundles between  $(0, 3)$  and  $(3, 0)$ . This is clearly a distortion of the truth.

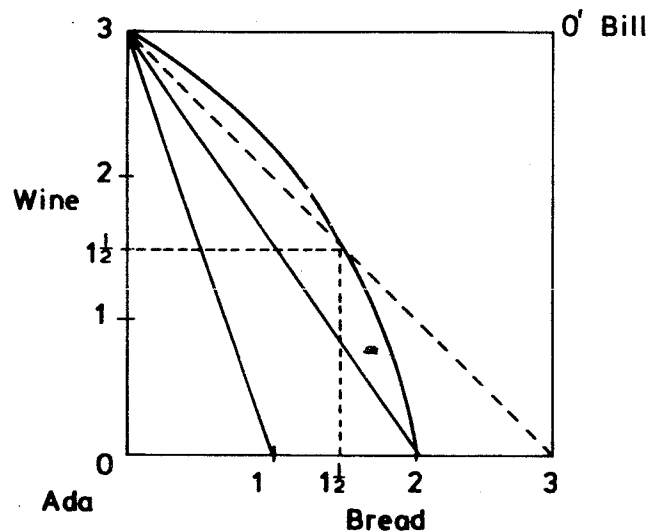


Fig. 1.23.

But it does give us back an 'equilibrium'. At equal prices Ada has the whole line from  $(0, 3)$  to  $(3, 0)$  as best choices and one of these  $(1\frac{1}{2}, 1\frac{1}{2})$  gives us an equilibrium (fig. 1.23). Thus we are back in the old situation having artificially added this line to Ada's offer curve.

Now this was a distortion of Ada's preferences. In effect we flattened out her indifference curves until they coincided with the price lines and were thus just convex. However, the real question is how important is this distortion? By changing the demand of individuals in the way we did we restore our ability to use the mathematical results mentioned earlier. Thus, we may prove the existence of 'equilibria'. How close to equilibria are these objects?

To see the answer consider augmenting the number of Adas. Now add their offers at each price. The problem occurred at equal prices for the two goods. At such prices the demands of the two look as in fig. 1.24(a). Clearly,

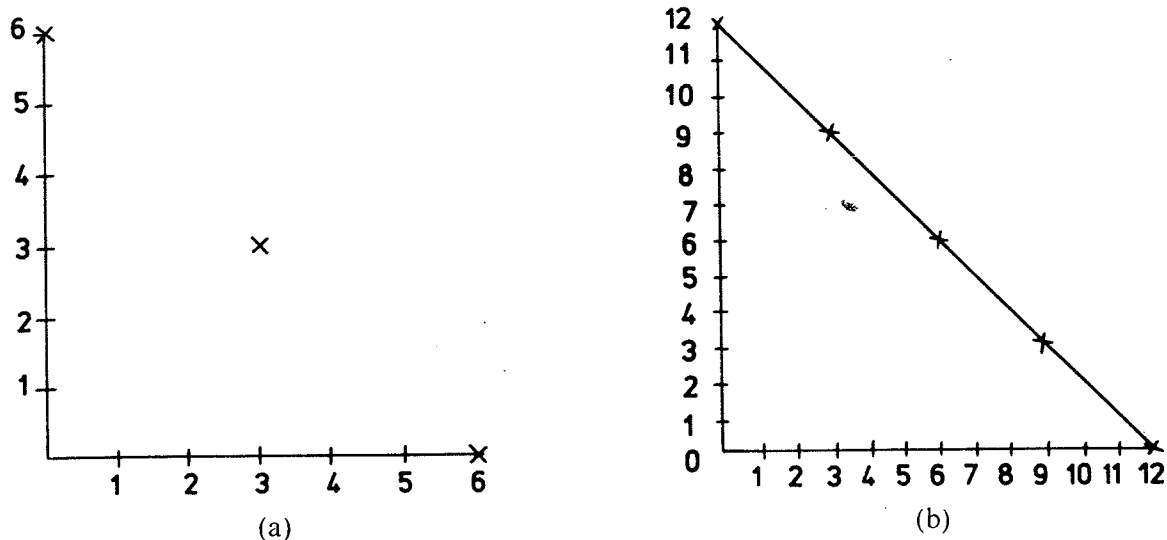


Fig. 1.24.

with two Adas and two Bills our problem is solved. If one Ada takes  $(3, 0)$  and the other  $(0, 3)$ , as best bundle for them at equal prices, and each Bill chooses  $(1\frac{1}{2}, 1\frac{1}{2})$  then we have an equilibrium. In this case, adding one Ada was enough to generate a point on the offer curve which coincided with one on Bill's offer curve. However, if we had had four Adas the picture would have been as in fig. 1.24(b). It is clear that the points are gradually filling out the gaps, thus reducing the discontinuity of the aggregate offer curve.

What we can and do show in ch. 6 is that if all individuals can have non-convex preferences then we can find an equilibrium by 'convexifying' their demand. Furthermore, if there are many individuals almost all of them will receive something in their original unmodified demand set at the 'equilibrium' prices. Thus, in a large economy non-convexity is relatively unimportant. Individual irregularity is not inconsistent with aggregate regularity.

## 11. Forewarned is forearmed

Having briefly summarized what is done in the book, in the context of the comfortable Ada-Bill world, we now move on to a more rigorous analysis of these questions. Hopefully, the insights obtained here will enable the reader to keep the basic themes in mind even when the orchestration, unavoidably, becomes a little dense!