An Equilibrium Analysis of the Simultaneous Ascending Auction

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Abstract

We analyze the dynamic simultaneous ascending auction (SAA), which was pioneered by the US Federal Communications Commission (FCC) in 1994 and has since become the standard to conduct large-scale, large-stakes spectrum auctions around the world. We consider an environment where local bidders, each interested in a single item, compete against one or more global bidders with super-additive values for combinations of items. In the SAA, competition takes place on an item-by-item basis, which creates an *exposure problem* for global bidders – when competing aggressively for a package, a global bidder may incur a loss when winning only a subset. We characterize the Bayes-Nash equilibria of the SAA, evaluate the impact of the exposure problem on revenue and efficiency, and compare its performance to that of the benchmark Vickrey-Clarke-Groves (VCG) mechanism. We show that individual and social incentives are aligned in the SAA in the sense that bidders' drop-out levels maximize expected welfare. Unlike the VCG mechanism, however, the SAA is not fully efficient because when a bidder drops out, information about others' values has been only partially revealed. Like the VCG mechanism, the SAA exhibits perverse revenue properties: due to the exposure problem, the SAA may result in non-core outcomes where local bidders obtain items at very low prices, and seller revenue can be decreasing in the number of bidders. Moreover, the SAA may result in *lower* revenues than the VCG mechanism. Finally, when the number of items grows large, the SAA and VCG mechanisms become (efficiency and revenue) equivalent.

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1. Introduction

In recent years, governments around the world have employed auctions to award licenses for the rights to operate in certain markets. The spectrum auctions conducted by the US Federal Communications Commission (FCC) provide a particularly prominent example. In the FCC auctions, telecom firms compete for blocks of frequencies (typically on the order of 10-20MHz) defined over certain geographic areas.[1] The auction format used by the FCC is the simultaneous ascending auction (SAA), which is a dynamic, multi-round format in which the items are put up for sale simultaneously and the auction closes only when bidding on all items has stopped. The SAA has become the standard to conduct large-scale, large-stakes spectrum auctions and has generated close to $80 billion for the US Treasury and hundreds of billions worldwide.

An important property of the SAA is that when items are substitutes and bidding is "straightforward," i.e. in each round of the auction bidders place minimum acceptable bids on those licenses that provide the highest current profits, then prices converge to competitive equilibrium prices and a fully efficient outcome results (Milgrom, 2000; Gul and Stacchetti, 2000). However, in many of the FCC auctions there are synergies between licenses for adjacent geographic regions, and bidders' values for combinations of licenses exceed the sum of individual license values. For example, the bid regressions reported by Ausubel, Cramton, McAfee, and McMillan (1997) show that the highest losing bid on a license is higher if the bidder who placed the bid has won or eventually wins a license. Bajari and Fox (2009) apply a structural econometrics model to data from FCC auction #5 and find evidence for substantial value complementarities: they estimate that the value of a nationwide package is 69% more than the sum of underlying values.[2] Value complementarities were considered even more important in the recently conducted FCC auction #73, where potential entrants, e.g. Google, competed against established incumbents such as Verizon and AT&T for highly valuable 700MHz spectrum.[3] Most experts believed that an entrant could have a viable business plan only if it would acquire a "national footprint," i.e. a set of licenses covering the entire United States.

In this paper, we consider an environment where one or more global bidders (entrants) have super-additive values for the licenses, i.e. for global bidders licenses are *complements* rather than substitutes. For this environment an often cited problem of the item-by-item competition

[1] The different frequency domains that have been put up for sale in the 73 FCC auctions since 1994 accommodate different usages, including wireless and cellular phone applications, mobile television broadcasting, and air-to-ground communication. See `http://wireless.fcc.gov/auctions/default.htm?job=auctions_all`.

[2] FCC auction #5 (also known as the "C-block auction") was conducted in 1995 and generated over $10 billion in revenue.

[3] FCC auction #73 was conducted in 2008 and generated a record $19 billion in revenue. It was the first combinatorial auction conducted by the FCC, based on hierarchically structured packages (Rothkopf, Pekec, and Harstad, 1998) and a novel pricing rule (Goeree and Holt, 2009).

that occurs under SAA is that global bidders face an *exposure problem* – when competing aggressively for a package, global bidders may incur a loss when winning only an inferior subset. Foreseeing the possibility of being "exposed," global bidders may decide to bid cautiously and drop out early, which could adversely affect the auction's revenue and efficiency.[4] For example, the counterfactual experiments conducted by Bajari and Fox (2009) demonstrate that in FCC auction #5 only 50% of the total available surplus was captured.[5] In addition, a substantial body of laboratory evidence documents the negative impact of the exposure problem on the SAA's performance (see, e.g., Brunner et al., 2009, and references therein).

Despite the potential shortcomings of the item-by-item competition underlying SAA, it has been the preferred choice for most spectrum auctions. Alternatives allowing for package bids were either considered too complex or thought to be prone to "free riding" (Milgrom, 2000).[6] Furthermore, the familiar Vickrey-Clarke-Groves (VCG) mechanism, which guarantees full efficiency even in the presence of value complementarities, is generally dismissed because of its perverse revenue properties. In particular, the VCG mechanism can lead to non-core outcomes that result in high bidder profits and low seller revenue. Moreover, seller revenue can decrease when more bidders participate. The following three-bidder, two-item example provided by Ausubel and Milgrom (2006) illustrates these shortcomings. Suppose local bidder 1 is interested only in item A, local bidder 2 is interested only in item B, the global bidder 3 is interested only in the package AB, and all bidders' values (for individual items or the package) are \$1 billion. The VCG mechanism assigns the items efficiently to bidders 1 and 2, but at zero prices![7] Besides generating the lowest possible revenue, this outcome is outside the core as the seller and global bidder can form a blocking coalition. Moreover, excluding one of the local bidders, raises the seller's revenue to \$1 billion. These perverse revenue properties, shown here in a complete-information setting, carry over to the Bayesian framework studied in this paper where bidders' values are private information.[8]

[4]Milgrom (2000) argues that the different per-unit-of-bandwidth prices observed for small and large licenses in the Dutch DCS-1800 auction reflect the exposure problem. A similar observation applies to the recent FCC auction #66, where 12 large (F-band) licenses providing 20MHz of nationwide coverage sold for \$4.2 billion while 734 small (A-band) licenses also providing 20MHz of nationwide coverage went for \$2.3 billion.

[5]Bajari and Fox (2009) show that surplus could have been doubled had the FCC offered large regional licenses or a nationwide package in addition to individual licenses.

[6]In package auctions, small bidders who drop out early ("free ride") may earn windfall profits when other small bidders remain active and outbid the global bidders. After all, a small bidder's concern is simply whether *as a group* they meet the threshold set by a global bidder's package bid. Of course, if all local bidders free ride this threshold may never be met, with adverse effects for the auction's revenue and efficiency – this is known as the *threshold problem*.

[7]Ausubel and Milgrom (2006) develop this example further in a theorem that shows that bidders' Vickrey payoffs are the highest payoffs over all points in the core.

[8]For instance, when bidders' values in the example above are uniformly distributed between 0 and 1, the revenue of the VCG mechanism is $\frac{1}{3}$ with only a single local bidder, $\frac{1}{4}$ with two local bidders, and $\frac{1}{10}$ with three

The goal of this paper is to compare the performance of the SAA to that of the VCG mechanism in a setting with complementarities. We provide a complete characterization of equilibrium bidding in the SAA, which allows us to determine the impact of the exposure problem on efficiency and revenue. Our setup is general in that we allow for arbitrary numbers of local and global bidders, arbitrary distributions of local and global bidders' values, and a general convex valuation function to capture global bidders' value complementarities.

For this general environment we prove that individual and socially optimal incentives are aligned in the SAA, as is the case in the VCG mechanism. In particular, bidders' drop-out levels, chosen to maximize their expected profits, also maximize expected welfare (efficiency). Unlike VCG, however, the SAA is not fully efficient because when a bidder drops out there is residual uncertainty about the values of other active bidders. We show that the efficiency gain of the VCG mechanism does not benefit the seller, however, but merely results in higher profits for the global bidders.

We also demonstrate that the SAA shares the poor revenue-generating features of the VCG mechanism. Due to the exposure problem the SAA can result in non-core outcomes characterized by low seller revenues – indeed, seller revenue in the SAA may be *less* than in the VCG mechanism. In addition, seller revenue in the SAA can be declining in the number of bidders.[9] Finally, the similarities between the SAA and VCG mechanisms become even stronger as the number of items grows, and the two mechanisms are (revenue and efficiency) equivalent in the limit.

Our findings contrast with those for the substitutes environment that typically is assumed to analyze the SAA (e.g. Milgrom, 2000). Since value complementarities are the rule rather than the exception, our results reinforce the interest of policy makers in further improving the design of auctions to award spectrum licenses, including the possibility of allowing for package bids.

1.1. Related Literature

Auctions in which bidders have synergistic values have often been analyzed within a complete-information setting, see, for instance, Szentes and Rosenthal (2003a,b). There are relatively few theoretical papers that apply the standard Bayesian framework of incomplete information. An early exception is Krishna and Rosenthal (1996) who study the simultaneous sealed-bid second-price auction (SSA). Similar to our bidding environment, local bidders in their setup

local bidders.

[9]For the aforementioned example (see footnote 8), SAA revenue is $\frac{1}{3}$ with a single local bidder, 0.27 with two local bidders, and 0.087 with three local bidders.

are interested in only one object while global bidders are interested in multiple objects for which they have synergistic values. Krishna and Rosenthal derive an explicit solution for the case of two items and show how it varies with the synergy level. They also discuss the extension to more than two items and provide a numerical comparison of revenue in alternative formats. Other papers that study the SSA include Rosenthal and Wang (1996), who allow for common values and partially overlapping bidder interests, and a more recent paper by Chernomaz and Levin (2008) who use theory and experiments to analyze the SSA and a package bidding variant when local bidders have identical values.

Ascending formats have been analyzed either assuming a clock price that rises in response to excess demand or assuming that bidders can name their own bids (i.e. submit any bid they want). In the latter category, Brusco and Lopomo (2002) demonstrate the possibility of collusive demand-reduction equilibria in the SAA. They find that increasing the number of bidders and objects narrows the scope for collusion. Brusco and Lopomo (2009) analyze the effects of budget constraints. Zheng (2008) shows that jump bidding may serve as a signaling device to alleviate the inefficiencies that result from the exposure problem. Albano, Germano, and Lovo (2006) analyze a ("Japanese style") clock version of the ascending auction (as we do in this paper) for a setting with only two items. They note the equivalence between the SAA and a "survival auction" and point out that many of the collusive or signaling equilibria that occur when bidders can name their bids do not arise for the clock variant of the SAA.

1.2. Organization

This paper is organized as follows. Section 2 provides an equilibrium analysis of the SAA for the case of one global bidder. Section 3 proves the social optimality of bidder's drop-out choices, and Section 4 extends the result to an arbitrary number of global bidders. Section 5 establishes the equivalence of the SAA and VCG mechanisms as the number of items grows large. Section 6 concludes.

2. The Simultaneous Ascending Auction

Consider an environment with $n \geq 1$ local bidders and $K \geq 1$ global bidders who compete for n items labeled $1, \ldots, n$. Local bidder i is interested only in acquiring item i, for which she has value v_i. The local bidders' values are identically and independently distributed according to $F(\cdot)$. Global bidder j's value for winning k items is $\alpha(k)V^j$, where the V^j are identically and independently distributed according to $G(\cdot)$, and $\alpha(k)$ is increasing in k with $\alpha(0) = 0$ and

$\alpha(n) = 1$.[10] We define the "marginal values" $V_k^j = (\alpha(n+1-k) - \alpha(n-k))V^j$ for $k = 1, \ldots, n$ so that global bidder j's marginal value of the first item is V_n^j, of the second item is V_{n-1}^j, ..., and of the n-th item is V_1^j.

Assumption 1 (complementarities). *For global bidder $j = 1, \ldots, K$, the marginal values form a non-decreasing sequence $V_n^j \leq V_{n-1}^j \leq \ldots \leq V_1^j$. Moreover, we say that*

(i) there are "no complementarities" when $V_n^j = \ldots = V_1^j = V^j/n$.

(ii) there are "extreme complementarities" when $V_n^j = \ldots = V_2^j = 0$ and $V_1^j = V^j$.

The simultaneous ascending auction (SAA) is modeled using n price clocks that tick upward (at equal and constant pace) when two or more bidders accept the current price levels. If at most one bidder accepts the new price then the price clock for that item stops and the bidder who accepted the higher price is allocated the item at the last price accepted by other bidders (ties are resolved randomly). Under this mechanism, local bidders have a dominant strategy to bid up to their values. A global bidder's strategy is complicated by the fact that when competing aggressively for a package, the global bidder may suffer a loss when she is able to win only an inferior subset. Foreseeing the possibility of being "exposed" and incurring a loss, the global bidder may decide to bid cautiously and drop out early, which could adversely affect the auction's revenue and efficiency — this is known as the *exposure problem.*

2.1. Single Global Bidder

We'll use the notation $B_k^K(V)$ to denote a global bidder's bidding function when K global bidders and k out of n local bidders are active. In this section we focus on the case of a single global bidder ($K = 1$). To derive the optimal strategy for the global bidder, suppose first that there is only one item for sale ($k = 1$). It will prove useful to introduce the notation $F_n(v_n|p) = 1 - ((1 - v_n)/(1 - p))^n$, which is the conditional probability that the minimum of n local bidders' values is less than v_n given that the minimum is no less than p.

If the current price level is p and the global bidder chooses a drop-out price level $B_1^1(V)$, her expected profits are

$$\Pi_1^1(V, p) \;=\; \int_p^{B_1^1(V)} (V_1 - v_1) dF_1(v_1|p),$$

with v_1 the local bidder's value. The integrand $\pi_1^1(V, v_1) = V_1 - v_1$ is the global bidder's profit if the local bidder drops out at price v_1. Clearly, the global bidder's expected profit is maximized by choosing the drop-out price $B_1^1(V)$ such that $\pi_1^1(V, B_1^1(V)) = 0$, or $B_1^1(V) = V_1$.

[10] The assumption that (local and global) bidders' values are identically distributed can easily be relaxed (at the cost of extra notation).

Next, consider the case $k = 2$. Suppose the current price level is p and the global bidder chooses a drop-out price level $B_2^1(V)$ (for both items), her expected profit is non-trivial only when the local bidder with the lower value drops out before $B_2^1(V)$. Once a local bidder drops out, the global bidder faces competition only in a single market and she is willing to bid up to V_1 in this market. The reason is that her profit for the item on which bidding stopped is sunk (i.e. independent of whether or not she wins an additional item). The global bidder's expected profit can be written as

$$\Pi_2^1(V,p) \; = \; \int_p^{B_2^1(V)} \Big\{ \int_{v_2}^{V_1} (V_1 - v_1) dF_1(v_1|v_2) + (V_2 - v_2) \Big\} dF_2(v_2|p),$$

where v_2 (v_1) denotes the lower (higher) of the local bidders' values. The integrand $\pi_2^1(V, v_2) = \int_{v_2}^{V_1}(V_1 - v_1)dF_1(v_1|v_2) + (V_2 - v_2)$ is the global bidder's expected profit conditional on the local bidder with the lower value dropping out at v_2. The first term arises when the global bidder wins the remaining item, i.e. when $v_1 \geq v_2$ is less than V_1 (since the global bidder bids up to V_1 for the remaining item) in which case the global bidder wins the additional item and pays v_1 for it. The second term indicates that the global bidder profits $V_2 - v_2$ from the item for which bidding stopped first, irrespective of whether she wins the additional item.

Again, the global bidder's optimal drop-out level follows from $\pi_2^1(V, B_2^1(V)) = 0$, which yields $B_2^1(V) = V_2 + \Pi_1^1(V, B_2^1(V))$. Note that the global's profit can be recursively expressed using the profit for the single local-bidder case: $\Pi_2^1(V,p) = \int_p^{B_2^1(V)}\{\Pi_1^1(V, v_2) + (V_2 - v_2)\}dF_2(v_2|p)$. This recursive relation can be generalized to the case of more than two items.

Proposition 1. *The global bidder's optimal drop-out level solves* $B_k^1(V) = V_k + \Pi_{k-1}^1(V, B_k^1(V))$, *where the payoffs satisfy the recursive relation*

$$\Pi_k^1(V,p) \; = \; \int_p^{B_k^1(V)} \{\Pi_{k-1}^1(V, v_k) + (V_k - v_k)\} dF_k(v_k|p), \tag{2.1}$$

with $\Pi_0^1(V,p) = 0$.

The proposition implies a set of fixed-point equations from which the optimal bids can be solved recursively: $B_1^1(V) = V_1$, $B_2^1(V) = V_2 + \int_{B_2^1(V)}^{B_1^1(V)}(V_1 - v_1)dF_1(v_1|B_2^1(V))$,

$$B_3^1(V) \; = \; V_3 + \int\limits_{B_3^1(V)}^{B_2^1(V)} \Big\{ \int\limits_{v_2}^{B_1^1(V)} (V_1 - v_1)dF_1(v_1|v_2) + (V_2 - v_2)\Big\} dF_2(v_2|B_3^1(V)),$$

etc. The intuition behind these bidding functions stems form a familiar break-even condition: when a license is marginally won at price $B_n^1(V)$, its value plus the expected payoffs from con-

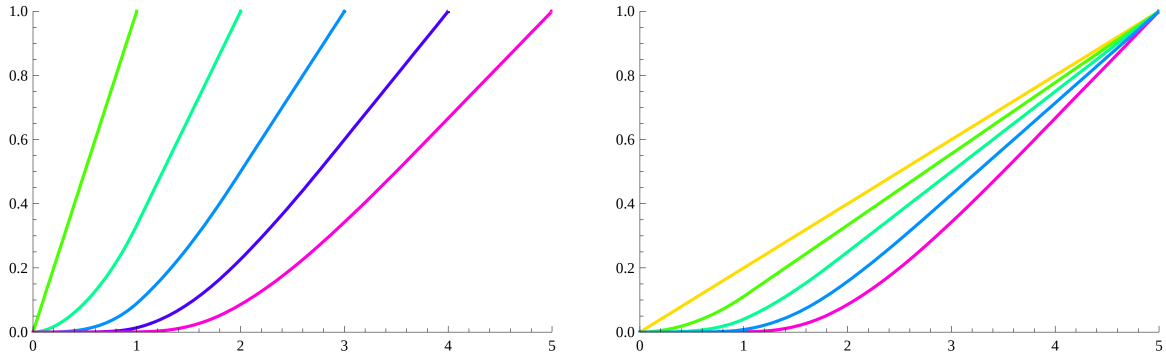

Figure 1. The Global Bidder's Optimal Drop-Out Level When Complementarities are Extreme: $B^1_n(V)$ (left) For $n = 1, \ldots, 5$ and $B^2_n(V)$ (right) For $n = 0, \ldots, 5$.

tinuing (knowing all remaining bidders' values exceed $B^1_n(V)$) must balance this cost.

Example 1. To illustrate, suppose $n = 2$ and the local bidders' values are uniformly distributed. We have $B^1_1(V) = V_1$ and a simple calculation shows that

$$B^1_2(V) \;=\; \frac{1}{3}\Big(1 + V_1 + V_2 - \sqrt{(2 - V_1 - V_2)^2 - 3(1 - \min(1, V_1))^2}\,\Big) \tag{2.2}$$

For the case of extreme complementarities ($V_2 = 0$, $V_1 = V$), these bidding functions are illustrated by the two left-most lines in the left panel of Figure 1 (the right panel will be discussed in Section 4). The left panel also shows the optimal bidding functions $B^1_n(V)$ for higher values of n.

This setup can be used to show the perverse revenue properties of the VCG mechanism and the SAA. Figure 2 displays, for different levels of the global bidder's value ($0 \leq V \leq 1.5$) and three local bidders ($n = 3$), the expected welfare, revenue, global bidder's profit, and local bidders' profits in the VCG (solid) and SAA (dashed) mechanisms. Comparing the top panels of Figure 2 shows that the difference in welfare between the two mechanisms is the same as the difference in the global bidder's profit. Likewise, comparing the bottom panels shows that the difference in revenue is equal to the difference in local bidders' profits. These results are not special to the uniform case – in the next section we prove them more generally. The bottom panels show that for low values of V, local bidders have higher profits in the SAA and seller revenue is lower as a consequence – the exposure problem causes the global bidder to drop out early, resulting in windfall profits for local bidders.[11] Note that both the SAA and VCG

[11] These results can be used to generate the revenue numbers of the example in the Introduction (see footnotes 8 and 9). If the global bidder's value for package is uniformly distributed on $[0, 1]$ (and complementarities are extreme so that the global's value for a single item is 0) then revenues of the VCG mechanism are $(\frac{1}{3}, \frac{1}{4}, \frac{1}{10})$ for $n = 1, 2, 3$ while revenues of the SAA are $(\frac{1}{3}, 0.27, 0.087)$.

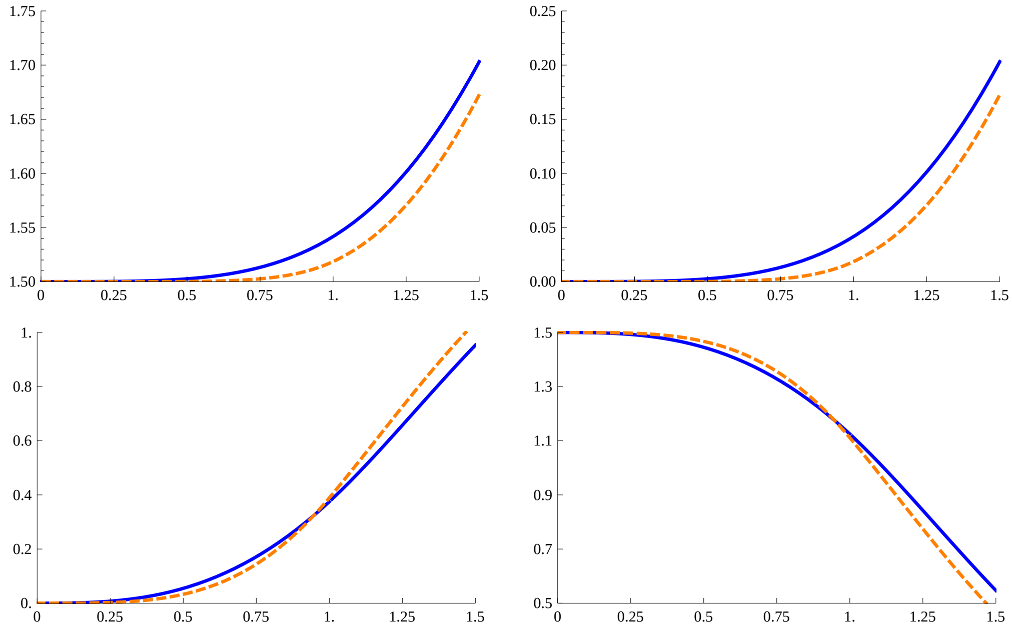

Figure 2. Welfare (top-left), Revenue (bottom-left), the Global Bidder's Profit (top-right), and the Local Bidders' Profits (bottom-right) for VCG (solid) and SAA (dashed) as Functions of the Global Bidder's Value $0 \leq V \leq 1.5$ with Three Local Bidders and Uniform Valuations.

generate non-core outcomes: when the global bidder's is $V = 0.4$, for example, the seller's revenue is close to 0 and the seller and global bidder could thus form a blocking coalition.

3. Social Optimality

The optimal drop-out levels of the global bidder shown in the left panel of Figure 1 illustrate the effects of the exposure problem in equilibrium. Consider, for instance, the case of five local bidders and suppose the global bidder is equally strong in expectation, i.e. the global bidder's value for the package is 2.5. When all five local bidders are active, the global bidder drops out when the price for each item is 0.2 (see the lowest line in the left panel of Figure 1), which means that the global bidder drops out at 40% of the package value! This does not necessarily mean, however, that efficiency is negatively affected. The lowest line in the left panel of Figure 1 only applies when all five local bidders are active, and if this occurs at an item price of 0.2 then the sum of the local bidders' expected values is 3 (not 2.5). Hence, efficiency may be improved when the global bidder drops out (especially when complementarities are extreme, as in Figure 1, and the global bidder derives no value from winning less than five items).

We next determine what a bidder's drop-out level would be if it were chosen to maximize expected welfare (rather than to maximize expected profits). When a bidder drops out, her information consists of her own value, the values of bidders that have already dropped out, and updated information about the values of bidders that are still active. In other words, the welfare function to be maximized is based on a mixture of "ex post" information (values of bidders that have dropped out) and "ex ante" information (values of active bidders). As a result, there are necessarily some inefficiencies in the SAA unlike the VCG mechanism where the welfare function is based on only "ex post" information about bidders' values.

Obviously, for a local bidder the socially optimal drop-out level is equal to her value. To derive the socially optimal drop-out level for the global bidder consider first the case of a single item ($k = 1$). The social planner would choose $B_1^1(V)$ to maximize

$$W_1^1(V, p) \;=\; \int_p^{B_1^1(V)} V_1 dF_1(v_1|p) + \int_{B_1^1(V)}^1 v_1 dF_1(v_1|p)$$

where the first (second) term corresponds to the global (local) bidder winning the item. Comparing the expression for welfare to the global's profit $\Pi_1^1(V, p)$ in Section 2.1 shows that $W_1^1(V, p) = \Pi_1^1(V, p) + E(v|v > p)$. In other words, welfare and the global bidder's profit differ only by a constant independent of $B_1^1(V)$. Hence, the drop-out level chosen by a profit-maximizing bidder maximizes welfare. We next generalize this to an arbitrary number of items.

Proposition 2. *Bidders' drop-out levels maximize expected welfare.*

Proof. We prove, by induction, that $W_k^1(V, p) = \Pi_k^1(V, p) + kE(v|v > p)$ for all $k \geq 1$. Above we have shown it is true for $k = 1$. For $k \geq 2$ we have:

$$\begin{aligned}
W_k^1(V, p) &= \int_p^{B_k^1(V)} W_{k-1}^1(V, v_k) dF_k(v_k|p) + \int_{B_k^1(V)}^1 \sum_{i=1}^k v_i \, dF_k(v_k|p) \\
&= \int_p^{B_k^1(V)} (\Pi_{k-1}^1(V, v_k) - v_k) dF_k(v_k|p) + \int_p^{B_k^1(V)} ((k-1)E(v|v > v_k) + v_k) dF_k(v_k|p) \\
&\quad + \int_{B_k^1(V)}^1 \sum_{i=1}^k v_i \, dF_k(v_k|p) \\
&= \Pi_k^1(V, p) + kE(v|v > p)
\end{aligned}$$

In the first line, the second term on the right side corresponds to the case where the global bidder drops out before the local bidder with the lowest value among the k active local bidders, in which case all remaining items are awarded to the local bidders. The first term corresponds to the case where the local bidder drops out first (at price level v_k), in which case the social planner optimizes the continuation welfare $W_{k-1}^1(V, v_k)$ with one fewer local bidder. In going from the first to the second line we used the induction hypothesis, and in going from the second to the third line we used the recursive property of the global bidder's profit, see Proposition

1. Since welfare and the global bidder's profit differ only by a constant, $B_k^1(V)$ is chosen in a socially optimal manner. *Q.E.D.*

As in the benchmark Vickrey-Clarke-Groves (VCG) mechanism, the individual and social incentives are aligned in the SAA. The difference is that in the VCG mechanism all values are revealed at once, and the social planner maximizes welfare (not expected welfare) to ensure 100% efficiency. In contrast, in the SAA, bidders' drop-out levels maximize expected welfare (where the expectation is with respect to the values of bidders that are still active), which precludes full efficiency.

The efficiency gain in the VCG mechanism does not benefit the seller, however, but only the global bidder. Let $W^{SAA}(V)$, $R^{SAA}(V)$, $\Pi^{SAA}(V)$, and $\pi^{SAA}(V)$ denote the expected welfare, expected revenue, expected global bidder's profit, and expected local bidders' total profit under the SAA mechanism, where the (ex ante) expectation is taken over local bidders' values only. Similar definitions apply with respect to the VCG mechanism.

Corollary 1. *The efficiency gain of the VCG mechanism accrues to the global bidder*

$$W^{VCG}(V) - W^{SAA}(V) \;=\; \Pi^{VCG}(V) - \Pi^{SAA}(V) \tag{3.1}$$

while differences in the seller's revenue accrue to the local bidders

$$R^{SAA}(V) - R^{VCG}(V) \;=\; \pi^{VCG}(V) - \pi^{SAA}(V) \tag{3.2}$$

Proof. Recall that $W^{SAA}(V) = W_n^1(V,0)$ and $\Pi^{SAA}(V) = \Pi_n^1(V,0)$ differ by $nE(v)$, see the proof of Proposition 2. Suppose the VCG mechanism assigns k of the n licenses to the global bidder for which she pays the opportunity cost, which is the sum of the k lowest local bidders' values. Let $\hat{\Pi}^{VCG}$ and $\hat{W}^{VCG}$ denote the global's profit and welfare respectively as a function of the entire profile of bidders' valuations: $\hat{\Pi}^{VCG} = \sum_{\ell=1}^{k}(V_{n-\ell+1} - v_{n-\ell+1})$ and $\hat{W}^{VCG} = \sum_{\ell=1}^{k} V_{n-\ell+1} + \sum_{\ell=k+1}^{n} v_{n-\ell+1}$, so $\hat{W}^{VCG} = \hat{\Pi}^{VCG}(V) + \sum_{k=1}^{n} v_k$. Taking expectations with respect to local bidders' value shows that the global's expected profit $\Pi^{VCG}(V)$ and expected welfare $W^{VCG}(V)$ differ by $nE(v)$. This establishes (3.1). The equality in (3.2) now follows from the "accounting identity" $R = W - \Pi - \pi$. *Q.E.D.*

As we show in Section 5, the differences between the VCG and SAA mechanisms vanish when the number of items grows large. First, we extend the optimality result of Proposition 2 to the case of more than one global bidder.

4. Multiple Global Bidders

A global bidder's optimal bidding function when there are multiple global bidders follows from the same 'break even' logic that underlies the result of Proposition 1. First, consider the case of $K = 2$ global bidders and suppose k out the n local bidders are still active: the optimal bid $B^2_k(V)$ is determined by requiring that at this price level the marginal costs and benefits of staying in a little longer (i.e. by bidding as of type $V + \epsilon$) cancel. There are two possible marginal events: one occurs when a local drops out, in which case the other global bidder has a value no less than V. Hence, the continuation profits for a global bidder with value V are zero in this case: $\Pi^2_{k-1}(V, B^2_k(V)) = 0$. Alternatively, the other global bidder drops out, in which case the continuation profits are given by $\Pi^1_k(V, B^2_k(V))$. Furthermore, the global bidder now wins all the $(n - k)$ items for which the local bidders had already dropped out at a price of $B^2_k(V)$ for each item. Finally, when there are $K \geq 3$ global bidders, the only non-vanishing marginal term results from $K - 1$ global bidders dropping out at the same time, which produces the same marginal equation as when $K = 2$.

Proposition 3. *The global bidder's optimal drop-out level satisfies $B^K_k(V) = B^2_k(V)$ for $K \geq 2$ and*

$$\Pi^1_k(V, B^2_k(V)) + \sum_{\ell=k+1}^{n} (V_\ell - B^2_k(V)) \;=\; 0, \tag{4.1}$$

where the $\Pi^1_k(V, p)$ satisfy the recursion relations of Proposition 1.

It is worthwhile pointing out a few cases: $k = 0$ corresponds to the case where all local bidders have dropped out two (or more) global bidders are active. We then have $B^2_0(V) = V/n$ since at a price of V/n the global bidder is indifferent between winning nothing and winning everything at that price. For $k = n$ we have $\Pi^1_n(V, B^2_n(V)) = 0$ so $B^2_n(V) = B^1_n(V)$. Likewise, for $k = n-1$ we have $\Pi^1_{n-1}(V, B^2_{n-1}(V)) + V_n = B^2_{n-1}(V)$, which implies $B^2_{n-1}(V) = B^1_n(V)$ (see Proposition 1). In other words,

$$B^2_n(V) \;=\; B^2_{n-1}(V) \;=\; B^1_n(V). \tag{4.2}$$

The intuition is that both $B^2_n(V)$ and $B^2_{n-1}(V)$ are determined by the marginal event where the global bidder wins her first item: $B^2_n(V)$ corresponds to the case where the other global bidder first drops out and then a local bidder, and $B^2_{n-1}(V)$ corresponds to the case where a local bidder first drops out and then the other global bidder. Under both scenarios, however, the marginal equation results from considering the cost and benefit of winning the first item, which yields $B^1_n(V)$ as the optimal drop-out level.

Example 1 (continued). The right panel of Figure 1 shows the global bidder's optimal drop-out levels $B^2_n(V)$ for $0 \leq n \leq 5$ when local bidders' values are uniform and complementarities are extreme (see also Example 1). Note that there are five (not 6) lines since $B^2_5(V) = B^2_4(V)$, see (4.2), and that $B^2_k(V) \leq B^1_{k+1}(V) \leq B^1_k(V)$ for all k.

As we show next, the ranking of the global bidders' optimal drop-out levels holds more generally. Competition from other global bidders aggravates the exposure problem and *lowers* a global bidder's optimal drop-out level: $B^2_k(V) \leq B^1_k(V)$ for all k. In fact, we can show something stronger.

Lemma 1. *$B^2_k(V) \leq B^1_{k+1}(V)$ for $k = 0, 1, \ldots, n-1$.*

Proof. For $k = n-1$, this follows from (4.2). To prove the lemma for $k \leq n-2$ note that when $B^2_k(V) = B^1_{k+1}(V)$ the left-side of (4.1) is equal to

$$\Pi^1_k(V, B^1_{k+1}(V)) + \sum_{\ell=k+1}^{n} (V_\ell - B^1_{k+1}(V)) \;=\; \sum_{\ell=k+2}^{n} (V_\ell - B^1_{k+1}(V)) \;\leq\; \sum_{\ell=k+2}^{n} (V_\ell - V_{k+1}) \;\leq\; 0$$

where the first equality follows since $\Pi^1_k(V, B^1_{k+1}(V)) = B^1_{k+1}(V) - V_{k+1}$ (see Proposition 1), the first inequality follows since $B^1_{k+1}(V) \geq V_{k+1}$, and the second inequality follows from Assumption 1. Since the left side of (4.1) is strictly decreasing in $B^2_k(V)$ the above inequality implies that $B^2_k(V) \leq B^1_{k+1}(V)$ for $k = 0, 1, \ldots, n-1$. *Q.E.D.*

The fact that global bidders are more cautious when facing competition from other global bidders does not hurt efficiency. On the contrary, it implies that global bidders who do not have the highest value, and who should therefore not win any items in the optimal allocation, drop out before local bidders that should win items in the optimal allocation do.

Proposition 4. *Bidders' drop-out levels maximize expected welfare for any $K \geq 1$.*

Proof. Consider the global bidder with the highest value, V, among all the global bidders and suppose in the optimal allocation this global bidder is assigned k^* items. Once other global bidders have dropped out, social optimality follows from Proposition 2, i.e. $B^1_k(V) > v_k$ for $k = n-k^*+1, \ldots, n$ and $B^1_k(V) < v_k$ for $k = 1, \ldots, n-k^*$. We need to show that all other global bidders drop out before $B^1_{n-k^*+1}$. This follows since for all $V' < V$ and $k = n-k^*+1, \ldots, n$, we have $B^K_k(V') = B^2_k(V') \leq B^1_{k+1}(V') \leq B^1_k(V') < B^1_k(V) \leq B^1_{n-k^*+1}(V)$. *Q.E.D.*

Efficiency in the SAA is lower than that of the VCG only because when a bidder who drops out she does not know the values of those that are active, and, hence, her optimal drop-out level maximizes expected welfare instead of (ex post) welfare as in VCG. This difference vanishes when the number of items grows large and local bidders drop out at a predictable rate governed by the distribution function $F(\cdot)$.

5. Large Auctions

In many of the applications of the SAA the number of items is very large, e.g. in some of the FCC spectrum auctions more than a thousand items were sold. In this section we show that the differences between VCG and SAA vanish in this limit. In particular, the SAA becomes fully efficient and generates the same profits for the bidders and the seller as the VCG mechanism.

Let V denote the highest of the global bidders' values and $v_n \leq \ldots \leq v_1$ denote the (ordered) local bidders' values. When the highest-value global bidder wins k of n items, welfare is

$$W(k,V) \;=\; \sum_{\ell=n-k+1}^{n} V_\ell + \sum_{\ell=1}^{n-k} v_\ell.$$

In the limit when $n \to \infty$ the sum of local bidders' values will diverge, and we assume that the highest of the global bidders' values diverges as well, i.e. $V = n\hat{V}$ where $\hat{V}$ is distributed according to $G(\cdot)$ with bounded support.[12] We can then normalize welfare and profits on a per-item basis. Suppose the highest-value global bidder wins a fraction κ of all items then normalized welfare is $W(\kappa,\hat{V}) = \lim_{n\to\infty} W(\kappa n, n\hat{V})/n$ and the global bidder's normalized value of winning a fraction κ of all items is $\mathcal{V}(\kappa) = \lim_{n\to\infty} \alpha(\kappa n)\hat{V}$. Assumption 1 implies that $\alpha(\cdot)$ is convex, and, hence, so is $\mathcal{V}(\cdot)$. To simply notation, below we simply write V instead of $\hat{V}$ to indicate the normalized value.

The welfare maximizing fraction of items assigned to the highest-value global bidder now follows from $W(V) \equiv \max_\kappa W(\kappa,V)$, or, equivalently,

$$W(V) \;=\; \max_{0\leq\kappa\leq 1} \mathcal{V}(\kappa) + \int_{F^{(-1)}(\kappa)}^{1} v dF(v), \tag{5.1}$$

where we used that in the limit when n grows large, $v_{(1-\kappa)n}$ is asymptotically normally distributed with mean $F^{(-1)}(\kappa)$ and variance of order $1/n$ (David and Nagajara, 2003). The solution to the maximization problem in (5.1) is denoted $\kappa^*(V) = \mathrm{argmax}(W(\kappa,V))$ so that $W(V) = W(\kappa^*(V),V)$.

In the SAA, local bidders drop out at a known rate, e.g. at price level p a total of $F(p)$ local bidders have dropped out. Suppose there is only one global bidder ($K = 1$). The global

[12] Otherwise the fraction of items that the global bidders win tends to zero in the limit as $n \to \infty$.

bidder's optimal strategy is to bid up to a level $B^1(V)$ that maximizes her per-item profit:

$$\Pi(V) = \mathcal{V}(F(B^1(V))) - \int_0^{B^1(V)} vdF(v).$$

Note that $\Pi(V) = W(F(B^1(V)), V) - E(v)$ so the global bidder's optimal drop-out level is simply $B^1(V) = F^{(-1)}(\kappa^*(V))$.

Next consider the case of multiple global bidders. First, let $K = 2$. The optimal drop-out level $B^2(V)$ follows by requiring that the marginal benefits and costs of staying in a little longer (by bidding as of type $V + \epsilon$) cancel. This deviation affects the outcome only when the rival global bidder drops out in between (with probability $\epsilon g(V)$), in which case the net benefit is

$$\mathcal{V}(F(B^2(V))) - B^2(V)F(B^2(V)) + \int_{B^2(V)}^{B^1(V)} (\mathcal{V}'(F(y)) - y)dF(y) = 0.$$

Here the first term reflects the value of the $F(B^2(V))$ items the global bidder wins when her rival drops out, the second term is how much she pays for them, and the third term is her continuation profit when she proceeds to win additional items by bidding up to $B^1(V)$. Integrating this last term and using the definition of $W(V)$ shows that $B^2(V)$ solves

$$B^2(V)F(B^2(V)) + \int_{B^2(V)}^{1} ydF(y) - W(V) = 0. \tag{5.2}$$

It is easily verified that the left side of (5.2) is strictly increasing in $B^2(V)$ so the solution is unique. We next show that $B^2(V) \le B^1(V)$. Evaluating the left side of (5.2) at $B^2(V) = B^1(V)$ yields

$$\begin{aligned} B^1(V)F(B^1(V)) - \mathcal{V}(F(B^1(V)) &= \kappa^*(V)F^{(-1)}(\kappa^*(V)) - \mathcal{V}(\kappa^*(V)) \\ &\ge \kappa^*(V)(F^{(-1)}(\kappa^*(V)) - \mathcal{V}'(\kappa^*(V))) \\ &= 0, \end{aligned}$$

where the equality in the first line follows form the definition of $B^1(V)$, the weak inequality in the second line follows from convexity of $\mathcal{V}$, and the equality in the third line follows from the first-order condition for $\kappa^*(V)$, see (5.1).[13] Since the left side of (5.2) is strictly increasing in $B^2(V)$ this implies that $B^2(V) \le B^1(V)$.

[13] To be precise, the equality in the third line holds only for interior solutions. To account for possible boundary solutions, note that for $\kappa^*(V) = 0$ the expression in the second line vanishes. Furthermore, if $\kappa^*(V) = F(B^1(V)) = 1$, then the global bidder's optimal drop-out level is determined solely by the event when other global bidders drop out and $B^1(V) = B^2(V) = \mathcal{V}(1)$.

Finally, when $K \geq 3$, the marginal equation that determines $B^K(V)$ follows by requiring that the marginal benefits and costs of staying in a little longer (e.g. by bidding as of type $V + \epsilon$) cancel. This deviation affects the outcome only when all rival global bidders drop out in between, and the resulting marginal equation is the same as when $K = 2$. In the next proposition, $G(Z|V)$ denotes the conditional distribution of a global bidder's value, Z, given that her value is less than V.

Proposition 5. *With a large number of items, the global bidders' optimal drop-out levels satisfy $B^1(V) = F^{(-1)}(\kappa^*(V))$, where $\kappa^*(V)$ maximizes the welfare in (5.1), and $B^K(V) = B^2(V) \leq B^1(V)$ for $K \geq 2$, where $B^2(V)$ solves (5.2). Moreover:*

$$\begin{aligned}
W^{SAA}(V) &= W^{VCG}(V) = W(V) \\
R^{SAA}(V) &= R^{VCG}(V) = \int_0^V W(Z)dG(Z|V)^{K-1} - \int_{B^1(V)}^1 (1 - F(v))dv \\
\Pi^{SAA}(V) &= \Pi^{VCG}(V) = \int_0^V (W(V) - W(Z))dG(Z|V)^{K-1} \\
\pi^{SAA}(V) &= \pi^{VCG}(V) = \int_{B^1(V)}^1 (1 - F(v))dv
\end{aligned}$$

Proof. The best global bidder wins $F(B^1(V)) = \kappa^*(V)$ items so welfare is maximized: $W^{SAA}(V) = W(V)$. To determine the best global bidder's profit note that she wins an optimal fraction of items $F(B^1(V))$, which she values at $\mathcal{V}(F(B^1(V)))$, and for which she pays

$$\int_0^V \{B^2(Z)F(B^2(Z)) + \int_{B^2(Z)}^{B^1(V)} ydF(y)\}dG(Z|V)^{K-1}$$

Here the first term in the integral corresponds to the items the best global bidder wins (all at once) when the second-best global bidder drops out at $B^2(Z)$, and the second term corresponds to the items she wins when local bidders subsequently drop out between $B^2(Z)$ and $B^1(V)$. Using (5.2), we can rewrite the global bidder's profit as stated in the proposition. (Note that for $K = 1$, the expression reduces to $W(V) - W(0) = W(V) - E(v)$.) Local bidders with values higher than $B^1(V)$ win an item at a price $B^1(V)$ and the total profits for local bidders as a group therefore are

$$\int_{B^1(V)}^1 (v - B^1(V))dF(v) = \int_{B^1(V)}^1 (1 - F(v))dv$$

where we used partial integration. The seller's revenue follows from $R = W - \Pi - \pi$. It is standard to verify the expressions for welfare, revenue, and profits for the VCG mechanism. *Q.E.D.*

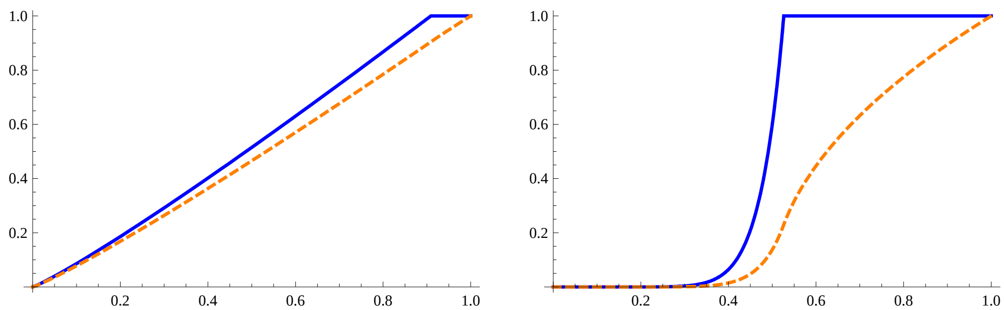

Figure 3. Global Bidder's Optimal Drop-Out Level with One (solid) or Two (dashed) Global Bidders and Weak (left) or Strong (right) Complementarities.

Example 2. Suppose value complementarities are parameterized as $\mathcal{V}(\kappa) = \kappa^\rho V$ with $1 \leq \rho \leq 2$ and a global bidder's value, V, is uniformly distributed on $[0, 1]$. A straightforward computation shows that

$$B^1(V) = \begin{cases} (\rho V)^{1/(2-\rho)} & \text{if } V \leq 1/\rho \\ 1 & \text{if } V \geq 1/\rho \end{cases}$$

and

$$B^2(V) = \begin{cases} \sqrt{\frac{2}{\rho} - 1}\,(\rho V)^{1/(2-\rho)} & \text{if } V \leq 1/\rho \\ \sqrt{2V - 1} & \text{if } V \geq 1/\rho \end{cases}$$

which are shown by the solid and dashed lines in Figure 3 for weak complementarities ($\rho = 1.1$) on the left and strong complementarities ($\rho = 1.9$) on the right. Note that the exposure problem gets worse as complementarities become stronger, causing low-value global bidders to drop out earlier and high-value global bidders to bid more aggressively.

6. Conclusions

We provide a general Bayes-Nash equilibrium analysis of the simultaneous ascending auction (SAA) when global bidders, with super-additive values for combinations of items, compete against smaller bidders interested in a single item. Due to the item-by-item competition in the SAA, global bidders face an *exposure problem* – when competing aggressively for the entire package, a global bidder may incur a loss when winning an inferior subset. The equilibrium analysis of this paper allows us to quantify the adverse effects of the exposure problem on efficiency and revenue of the SAA, and compare its performance to that of the benchmark Vickrey-Clarke-Groves (VCG) mechanism.

Our setup is very general in that it allows for arbitrary numbers of local and global bidders, arbitrary distributions of local and global bidders' values, and a general convex valuation function to capture global bidders' value complementarities. For this general environment we prove that individual and social optimal incentives coincide in the SAA, as is the case in the VCG mechanism. In particular, bidders' drop-out levels, which follow from profit-maximizing behavior, maximize expected welfare. Unlike VCG, however, the SAA is not fully efficient because at the time a bidder drops out, she does not know the values of other active bidders. Consequently, bidders' drop-out levels maximize expected welfare, not welfare (as in the VCG mechanism).

Importantly, our equilibrium analysis demonstrates that the exposure problem results in perverse revenue properties of the SAA. In particular, the SAA can easily lead to non-core outcomes in which local bidders obtain the items at very low prices. Moreover, the seller's revenue may decline as more bidders enter the auction. These shortcomings, which are well known for the VCG mechanism (e.g. Ausubel and Milgrom, 2006), were hitherto not known for the SAA simply because a general equilibrium analysis did not exist. The similarity between the SAA and VCG mechanisms becomes even more pronounced in larger auctions: when the number of items grow large, as is the case in many FCC spectrum auctions, SAA and VCG yield identical profits for the bidders and the seller.

The low-revenue outcomes the SAA produces in an environment with complementarities contrast with its superior properties when licenses are substitutes (e.g. Milgrom, 2000). Revenue of the SAA can even be less than that of the VCG mechanism, which is generally dismissed for giving too much surplus to bidders and leaving too little for the seller (e.g. Ausubel and Milgrom, 2006). Our findings demonstrate the adverse effects the exposure problem can have in item-by-item formats, and reinforce the interest of policy makers in more flexible auction institutions that accommodate bidders' synergistic preferences.

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