On the Impossibility of Core-Selecting Auctions

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November 10, 2009

Abstract

When goods are substitutes, the Vickrey auction produces efficient, core outcomes that yield competitive seller revenues. In contrast, with complements, the Vickrey outcome, while efficient, is not necessarily in the core and revenue can be very low. Non-core outcomes may be perceived as unfair since there are bidders willing to pay more than the winners' payments. Moreover, non-core outcomes render the auction vulnerable to defections as the seller can attract better offers afterwards. To avoid instabilities of this type, Day and Raghavan (2007) and Day and Milgrom (2007) have suggested to adapt the Vickrey pricing rule. For a simple environment with private information, we show that the resulting auction format yields lower than Vickrey revenues and inefficient outcomes that are on average further from the core than Vickrey outcomes. More generally, we prove that the Vickrey auction is the unique core-selecting auction. Hence, when the Vickrey outcome is not in the core, no individually-rational, incentive-compatible, core-selecting auction exists. Our results further imply that the competitive equilibrium cannot be implemented when goods are not substitutes. Moreover, even with substitutes, the competitive equilibrium can only be implemented when it coincides with the Vickrey outcome.

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1. Introduction

Practical auction design is often complicated by institutional details and legal or political constraints. For example, using bidder-specific bidding credits or reserve prices may be considered discriminatory and unlawful in some countries making it impossible to implement an optimal auction design. More generally, the use of sizeable reserve prices may cause political stress due to the fear that it slows down technological progress when licenses remain unsold. While constraints of this nature are common and important in practice, mechanism design theory has typically treated them as secondary to incentive constraints.

Recent work by Day and Raghavan (2007), Day and Milgrom (2008), and Day and Cramton (2008) breaks with this tradition and asks how close incentive constraints can be approximated if other (institutional or political) constraints are imposed first. In particular, these authors have proposed an alternative payment rule to fix some drawbacks of the well-known Vickrey-Clarke-Groves mechanism, or "Vickrey auction" for short. The Vickrey auction produces core outcomes that yield competitive seller revenues when goods are substitutes. However, when goods are complements, the Vickrey outcome, while efficient, is not necessarily in the core and seller revenue can be very low as a result. Non-core outcomes are "unfair" in that there are bidders willing to pay more than the winners' payments, which makes the auction vulnerable to defections as the seller can attract better offers afterwards. The low revenue, perceived unfairness, and instability of Vickrey outcomes can create legal and political problems, which the alternative payment rule seeks to avoid.

BCV payments[1] are such that (i) given reported bids, no group of bidders can block the outcome, (ii) bidders' profits are maximized, and (iii) payments are as close as possible to the original Vickrey payments. These properties ensure that the payments are (i) considered "fair," (ii) minimize the maximum possible gains from deviating from truthful bidding, and (iii) are unique. The requirement that payments are as close of possible to the original Vickrey payments furthermore reinforces the idea that the distortion of incentives is as small as possible.[2]

[1]BCV stands for bidder-optimal, "core-selecting," Vickrey-nearest. Here "core selecting" reflects the property that no coalition of bidders can block the outcome based on reported bids. It does *not* imply that BCV payments produce core outcomes (with respect to bidders' preferences) unless reported bids are always truthful. Below we show that this is not the case.

[2]It is important to point out that bidders' maximal incentives for (possibly large) deviations are

The introduction of BCV prices comes at the expense of (computational) complexity. When there are W winning bidders the number of non-blocking constraints to be verified can be as large as $2^W - 1$. And the checking of *each* constraint involves solving an NP-hard winner-determination problem. Day and Raghavan (2007) propose an ingenious algorithm in which constraints are generated "on the fly," i.e only the maximally-violated constraint is generated, which can greatly reduce the number of constraint checks. Nevertheless, one worries about the feasibility of finding BCV prices in applications with a sizeable number of bidders and items, especially in the "worst case" when many constraints are violated. On a positive note, an algorithm to find BCV prices will not "search in vain" (because such prices do exist[3]) nor will it produce "ambiguous" answers (because such prices are unique) – it may just take a long time.

Computational complexity issues are not the topic of this paper. Rather we study how well incentive constraints are "approximated" in the BCV auction. For complete-information environments it has been shown that the BCV auction yields the Vickrey outcome when it is in the core and results in higher seller revenues when it is not. Moreover, the BCV auction minimizes the maximal gain from deviating from truthful bidding (e.g. Day and Milgrom, 2007). These positive results, however, rely crucially on the assumption that bidders' values and, hence, their bids are commonly known. In most practical applications, bidders' values constitute proprietary information.

We first consider a simple incomplete-information environment where two local bidders interested in single items compete with a global bidder interested in the package. Bidders' values are privately known and uniformly distributed. We show that the BCV auction results in lower than Vickrey revenues and in inefficient outcomes that are on average *further* from the core than Vickrey outcomes. The reason for these negative results is that bidders no longer have a dominant strategy to bid truthfully. Indeed, we prove that bid shading is a "dominant property," i.e. even if others bid truthfully it is better to shade one's bid.

The natural follow-up question is whether there is any mechanism that can guarantee core outcomes, and we show that the answer is negative. We first prove that *any* core-selecting auction must be equivalent to the Vickrey auction in the sense that, for every

minimized. Erdil and Klemperer (2009) show that bidders always have marginal incentives to deviate from truthful bidding, and they propose the introduction of reference prices to reduce bidders' incentives for such marginal deviations.

[3]E.g. winning bidders paying their bids constitutes an outcome that cannot be blocked.

possible valuation profile, the seller's revenue and bidders' profits are identical. For the case of substitutes it is well known that the Vickrey outcome is in the core, and our results imply that the revenue from any core-selecting mechanism equals the Vickrey revenue in this case. But when the Vickrey outcome is not in the core, our results imply there exists *no* individually-rational, incentive-compatible mechanism that produces core outcomes. The intuition is that bidders' information rents preclude higher than Vickrey revenues and when the Vickrey revenue is so low that the outcome is not in the core, no core-selecting auction exists.

This paper is organized as follows. The next section introduces the bidding environment and explains the construction of BCV prices. In Section 3 we analyze the Bayes-Nash equilibrium of the BCV auction and evaluate its performance in terms of revenue and efficiency. In Section 4 we prove that core-selecting auctions are generally not possible. Section 5 concludes.

2. BCV Prices: An Example

Consider an environment with two local bidders and a single global bidder who compete for two items labeled A and B. Local bidder 1 (2) is interested only in acquiring item A (B), for which she has value v_1 (v_2). The global bidder is interested only in the package AB consisting of both items, for which she has value V.

In the Vickrey auction, bidders simply bid their values for the object they are interested in: $b_i(v_i) = v_i$ and $B(V) = V$. This yields a fully efficient outcome, i.e. the global bidder wins all items iff $B > \sum_{i=1}^{n} b_i$, or equivalently, iff $V > \sum_{i=1}^{n} v_i$. While the Vickrey auction generates full efficiency, it is well known that it may result in low revenues when the outcome is not in the core (e.g. Ausubel and Milgrom, 2006).

Lemma 1. *In the Vickrey auction, the outcome is always in the core when the global bidder wins but the outcome is never in the core when the local bidders win.*

Proof. When the global bidder wins, she pays the sum of the local bidders' bids so the local bidders cannot form a blocking coalition. When the local bidders win, the Vickrey payment for local bidder 1 is $p_1^V \equiv \max(0, B - b_2)$, i.e. the negative externality local bidder 1 poses on others, which is zero if the other local bidder wins even without local bidder 1. Similarly, $p_2^V \equiv \max(0, B - b_1)$. When either $b_1 > B$ or $b_2 > B$, or both, then

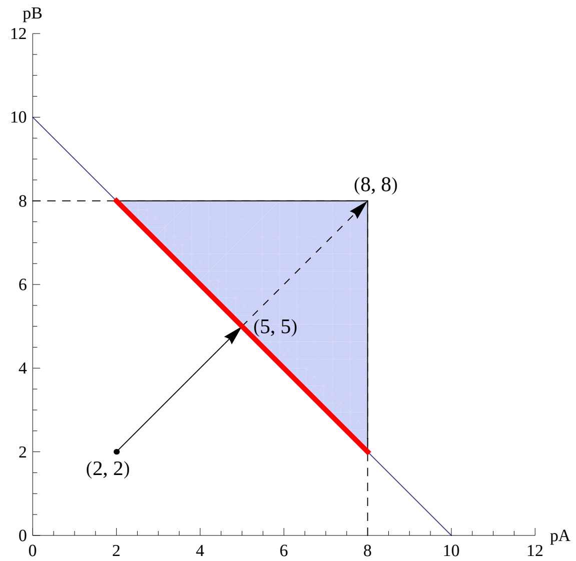

Figure 1. Constructing BCV prices, (5, 5), from Vickrey prices (2, 2). Pay-your-bid prices correspond to (8, 8).

the sum of payments is obviously less than the global bidder's B, and when both $b_1 < B$ and $b_2 < B$ but $b_1 + b_2 > B$, the sum of the payments is $2B - (b_1 + b_2) < B$. To summarize, when the local bidders win the outcome is not in the core because the coalition of the global bidder and the seller can block the outcome. *Q.E.D.*

When the local bidders win the auction the total sales price is less than the global bidder's bid. To remedy this flaw, various authors including Day and Raghavan (2007), Day and Milgrom (2008), and Day and Cramton (2008) have proposed to adjust Vickrey payments so that the outcome cannot be blocked by any coalition. In particular, in the BCV auction, winning local bidders' payments are adjusted so that they add up to exactly B and are as close as possible to Vickrey payments.[4]

The construction of BCV prices is illustrated in Figure 1 for the case of two local bidders with bids of 8 for the single items and one global bidder with a bid of 10 for the

[4] In other words, local bidder i's payment is perturbed, $p_i = p_i^V + \varepsilon_i$, where the $\varepsilon_i \geq 0$ follow from the quadratic programming problem:

$$\min\,(\varepsilon_1^2 + \varepsilon_2^2) \quad \text{s.t.} \;\; \varepsilon_1 + \varepsilon_2 = B - p_1^V - p_2^V$$

For our setup, the solution is simply $\varepsilon_1 = \varepsilon_1 = (B - p_1^V - p_2^V)/2$, so that $p_1 + p_2 = B$ and the outcome is in the core with respect to reported bids. It is interesting to note the similarity with the RAD mechanism proposed by Kwasnica et al. (2005) where linear prices are perturbed to satisfy competitive-equilibrium constraints, rather then core constraints.

package. The local bidders win and each pays the opportunity cost of $10-8=2$, as shown by the Vickrey point labeled "(2, 2)" in Figure 1. The set of prices that correspond to allocations that cannot be blocked are indicated by the shaded area. This area is bounded by the constraints $p_A \leq 8$ and $p_B \leq 8$ since local bidders cannot be made to pay more than their bids, and $p_A + p_B \geq 10$ to ensure that the losing global bidder and seller cannot form a blocking coalition. In particular, in the BCV auction, the winning local bidders' payments are chosen such they add up to exactly the global bidder's losing bid of 10. In Figure 1, this is indicated by the thick line: any point on this line minimizes the sum of local bidders' payments subject to the constraint that the outcome cannot be blocked. The final step is to generate a unique answer, which is accomplished by finding the unique point on the thick line that is closest to the original Vickrey prices (2, 2), i.e. the point (p_A, p_B) that minimizes the Euclidean distance $((p_A-2)^2+(p_B-2)^2)^{1/2}$. In the example, this yields the point labeled "(5, 5)" in Figure 1. Finally, pay-your-bid prices are shown by the point labeled "(8, 8)."

Note that the above construction subsumes that the profile of bids is fixed (i.e. bids are treated as "exogenous data") without addressing how changes in the payment rule may affect submitted bids. This approach would be valid in a complete-information framework where bidders' values, and, hence, their bids, are common knowledge. But in most applications, bidders' values constitute proprietary information.

3. A Bayesian Analysis

In the Vickrey auction, bidders have a dominant strategy to bid their values. Furthermore, when the global bidder wins, the outcome is always in the core with respect to submitted bids (see Lemma 1) so there are no adjustments to the global bidder's payment in the BCV auction. As a result, the global bidder's strategy is unaffected, i.e. truthful bidding remains optimal. For the local bidders truthful bidding is no longer optimal, however, since their own bids affect their payments.

The next proposition shows the degree to which local bidders "shade" their bids in response to the change in payment rule. For ease of exposition, we assume that all distributions are uniform, leaving the general case for the next section. In particular, the local bidders' values are uniformly distributed on $[0,1]$ and the global bidder's value for the package is uniformly distributed on $[0,2]$.

Proposition 1. *The Bayes-Nash equilibrium of the BCV auction is given by* $B(V) = V$ *and*

$$b(v) \;=\; \max(0, v - \alpha) \tag{3.1}$$

where $\alpha = \frac{1}{2}E(b(v)) = 3 - 2\sqrt{2}$.

Proof. Local bidder 1's optimal bid, $b(v_1)$, can be determined from the requirement that the cost and benefit of a marginal deviation cancel. Suppose local bidder 1 with value v_1 acts as if her type is $v_1 + \epsilon$, i.e. she deviates to a bid of $b(v_1) + \epsilon b'(v_1)$. The expected gain of this deviation occurs when it turns the local bidder from a loser to a winner, i.e. when the global bidder's value lies between $b(v_1) + b(v_2)$ and $b(v_1) + b(v_2) + \epsilon b'(v_1)$. Since the distribution of the global bidder's value is uniform on $[0, 2]$, this probability is equal to $\frac{1}{2}\epsilon b'(v_1)$. Also note that since the global bidder's bid and the sum of the local bidders' bids are equal (up to order ϵ) in this case, bidder 1's Vickrey payment is simply $V - b(v_2) = b(v_1)$, i.e. her own bid. Hence, the expected gain of the deviation is

$$\tfrac{1}{2}\epsilon(v_1 - b(v_1))b'(v_1). \tag{3.2}$$

To determine the expected cost of the deviation, note that local bidder 1's payment in the BCV auction is given by

$$p_1 \;=\; \max(0, B - b_2) + \tfrac{1}{2}\big(B - \max(0, B - b_1) - \max(0, B - b_2)\big),$$

(and bidder 2's payment is defined similarly). The only term affected by an increase in bidder 1's bid is the $\frac{1}{2}\max(0, B - b_1)$ term. Hence, when bidder 1 raises her bid by $\epsilon b'(v_1)$ her payment goes up by $\frac{1}{2}\epsilon b'(v_1)$ if and only if the global bidder's value is greater than $b(v_1)$ and less than $b(v_1) + b(v_2)$ (since otherwise the local bidders do not win). So the expected cost of local bidder 1's deviation is simply

$$\tfrac{1}{2}\epsilon b'(v_1)\int_0^1\int_{b(v_1)}^{b(v_1)+b(v_2)} \tfrac{1}{2}dV\,dv_2 \;=\; \tfrac{1}{4}\epsilon b'(v_1)\,E(b(v)). \tag{3.3}$$

The requirement that the expected gain (3.2) cancels the expected cost (3.3) yields the following first-order condition for the local bidders' optimal bids:

$$b'(v)\big(v - b(v) - \tfrac{1}{2}E(b(v))\big) \;=\; 0. \tag{3.4}$$

The unique solution is $b(v) = \max(0, v - \alpha)$ where $\alpha \equiv \frac{1}{2}E(b(v))$ follows from the fixed-point condition

$$\alpha \;=\; \tfrac{1}{2}\int_{\alpha}^{1}(v-\alpha)dv,$$

which yields $\alpha = 3 - 2\sqrt{2}$. *Q.E.D.*

Proposition 1 shows that the introduction of BCV prices creates incentives for local bidders to "free ride," i.e. each local bidder wants to win but prefers other local bidders to bid high. The consequences for the auction's performance are easy to calculate.

Corollary 1. *Compared to the Vickrey auction, the BCV auction reduces efficiency, reduces revenue, and produces outcomes that are further away from the core.*

Proof. A direct computation shows that the relative efficiency of the core-selecting auction is: $E_{BCV}/E_V = 98\%$, relative revenue is: $R_{BCV}/R_V = 91\%$, and the relative average Euclidean distance to the core is: $d_{BCV}/d_V = 126\%$. *Q.E.D.*

The main insight of Proposition 1, i.e. that truthful bidding is not an equilibrium in the BCV auction, holds under more general conditions.[5] Importantly, the proof that bidders should deviate from truthful bidding and shade their bids does *not* rely on an equilibrium argument. The next proposition shows that bid shading is a "dominant property," i.e. even when others bid truthfully it is optimal to shade one's bid.

Proposition 2. *Suppose the other local bidder bids according to* $b(v) = \max(0, v - \hat{\alpha})$ *and the global bidder bids according to* $B(V) = V$*, then the best response is given by*

$$b(v) \;=\; \max\big(0, v - \frac{1}{4}(1-\hat{\alpha})^2\big). \tag{3.5}$$

In other words, bidders should (maximally) shade their bids more if others bid truthfully.

Proof. A direct computation shows that a local bidder's expected payoff is

$$\pi(v,b) \;=\; \frac{1}{2}\big(v - \frac{1}{2}b\big)\big(b + \frac{1}{2}(1-\hat{\alpha})^2\big)$$

[5]For general distributions of the local bidders' values, $F(v)$, the optimal bid function is given by $b(v) = \max(0, v-\alpha)$ where now $\alpha = \frac{1}{2}\int_{\alpha}^{1}(v-\alpha)dF(v)$. For some distributions, the resulting performance measures of the BCV auction are *worse* than those of Corollary 1. Likewise, the assumption that the global bidder's value is uniformly distributed can be relaxed. The resulting bidding function for the local bidders is no longer linear but free riding still occurs.

when the other local bidder bids according to $b(v) = \max(0, v - \hat{\alpha})$ and the global bidder bids according to $B(V) = V$. Maximizing with respect to b yields (3.5). *Q.E.D.*

The fact that the BCV auction does not produce core outcomes with respect to bidders' preferences, raises the question whether there is some other mechanism that can achieve core outcomes. We next prove that the answer is negative.

4. An Impossibility Result

To glean some intuition, we first continue the above example with two local bidders and a global bidder with uniform valuations and then generalize to arbitrary environments. Since core allocations are efficient, the total surplus of a core allocation is given by

$$W(v_1, v_2, V) \;=\; \max\big(v_1 + v_2, V\big),$$

which is divided in seller revenue, R, local bidders' profits, π_i, and the global bidder's profit, Π:

$$W(v_1, v_2, V) \;=\; R(v_1, v_2, V) + \pi_1(v_1, v_2, V) + \pi_2(v_1, v_2, V) + \Pi(v_1, v_2, V).$$

The constraint that the allocation is in the core implies that

$$R(v_1, v_2, V) \;\geq\; \min\big(v_1 + v_2, V\big),$$

and, using $|x - y| = \max(x, y) - \min(x, y)$, we get the inequality

$$|v_1 + v_2 - V| \;\geq\; \pi_1(v_1, v_2, V) + \pi_2(v_1, v_2, V) + \Pi(v_1, v_2, V).$$

In particular, by taking an expectation over bidders' values we obtain the following necessary condition for core allocations

$$\int_0^2\int_0^1\int_0^1 |v_1 + v_2 - V| dv_1 dv_2 \frac{1}{2} dV \;\geq\; 2\int_0^1 \overline{\pi}(v)dv + \int_0^2 \overline{\Pi}(V)\frac{1}{2}dV \tag{4.1}$$

where $\overline{\pi}(v)$ and $\overline{\Pi}(V)$ denote the expected profits of a local and global bidder respectively. Using a standard envelope-theory argument, these expected profits can be calculated as

$$\overline{\pi}(v) \;=\; \int_0^v\int_0^2\int_0^1 \mathbf{1}_{V\le t+w}\,dw\frac{1}{2}dV dt \;=\; \frac{1}{4}\big(v+v^2\big)$$

and

$$\overline{\Pi}(V) \;=\; \int_0^V\int_0^1\int_0^1 \mathbf{1}_{T>v+w}\,dvdwdT \;=\; \begin{cases} \frac{1}{6}V^3 & 0\le V\le 1 \\ \frac{1}{6}(2-6V+6V^2-V^3) & 1\le V\le 2 \end{cases}$$

where we used that any core-selecting auction is efficient and that bidders' with the lowest possible values (of zero) have zero expected profits. It is straightforward to compare the left-side of (4.1), which equals 7/12, to the bidders ex ante expected profits on the right side, which equal 5/24 for each of the two local bidders and 7/24 for the global bidder. Since $7/12 < 2\cdot 5/24+7/24$ the necessary condition for core allocations in (4.1) is violated.

We next show how to generalize to arbitrary settings using the insights of the example. Suppose there are $N\ge 1$ bidders who are competing for $M\ge 1$ goods. Bidder i's private information, or type, is denoted by t_i, which is distributed according to $F_i(\cdot)$ with support $[\underline{t}_i,\overline{t}_i]$. So bidders' types are one-dimensional and independently distributed. Bidder i's private valuation for bundle x is given by $v_i(x,t_i)$, which is continuously differentiable in t_i. We assume quasi-linear preferences, i.e. when bidder i wins bundle x and pays an amount p, her profit is $\pi_i = v_i(x,t_i)-p$. The seller's profit, or revenue, is denoted R. Finally, for any subset of bidders $S\subseteq N$ we define the coalitional value

$$V_S \;=\; \max_{x\in X}\sum_{i\in S} v_i(x,t_i) \tag{4.2}$$

where X is the set of all feasible allocations. In words, V_S is the maximum possible value when the goods are allocated only to bidders in S.

First, consider the Vickrey auction, which is efficient so $R^V+\sum_{i\in W}\pi_i^V = V_N$. Note that the sum is only over the set of winning bidders, W, since losing bidders' profits are zero. Taking expectations we have[6]

$$\int\big(V_N-R^V\big)dF(t) \;=\; \sum_{i\in W}\int \pi_i^V dF(t) \;=\; \sum_{i\in W}\int (V_N-V_{N\setminus i})dF(t)$$

[6]Here $\int dF(t)$ is short for $\int_{\underline{t}_1}^{\overline{t}_1}\cdots\int_{\underline{t}_N}^{\overline{t}_N} dF_1(t_1)\cdots dF_N(t_N)$.

where we used $\pi_i^V = v_i - (V_{N\backslash i} - (V_N - v_i))$, i.e. bidder i's Vickrey payment is the negative externality imposed on others. We assume that the Vickrey auction results in zero expected profits for bidders with the lowest possible types, i.e. $\pi^V(\underline{t}_i) = 0$ for all i.[7]

Myerson's Lemma (see, for instance, Milgrom, 2004, p. 75) implies that bidders' expected profits in any other efficient mechanism differ from Vickrey expected profits only by a constant, i.e.

$$\sum_{i \in W} \int \pi_i dF(t) = \sum_{i \in W} \pi_i(\underline{t}_i) + \sum_{i \in W} \int \pi_i^V dF(t) \tag{4.3}$$

where individual rationality dictates that $\pi_i(\underline{t}_i) \geq 0$ for all i.[8] Since any core-selecting mechanism has to be efficient, we thus have

$$\int \big(V_N - R\big) dF(t) = \sum_{i \in W} \int \pi_i dF(t) = \sum_{i \in W} \int (V_N - V_{N\backslash i}) dF(t) + \sum_{i \in W} \pi_i(\underline{t}_i) \tag{4.4}$$

We will next combine equality (4.4) with a subset of all core constraints.

There are $2^N - 1$ core constraints that follow by considering all non-empty subsets of bidders $S \subseteq N$:

$$R + \sum_{i \in S \cap W} \pi_i \geq V_S$$

where we used that losing bidders' profits are zero. We first focus on $|W|$ constraints, one for each winning bidder. In particular, for each $i \in W$, let $S = N\backslash i$ in the previous inequality:

$$R + \sum_{j \in W \backslash i} \pi_j \geq V_{N\backslash i} \tag{4.5}$$

Summing over all $i \in W$ we have

$$|W|R + (|W| - 1)\sum_{i \in W} \pi_i \geq \sum_{i \in W} V_{N\backslash i}$$

A core allocation is efficient, $V_N = R + \sum_{i \in W} \pi_i$, which together with the previous inequality yields:

$$V_N - R \leq \sum_{i \in W} (V_N - V_{N\backslash i}) \tag{4.6}$$

[7] E.g., when the private values $v_i(\cdot, \underline{t}_i)$ are zero.

[8] The assumptions required for Myerson's Lemma to hold are that (i) bidders' types are one-dimensional, (ii) bidders' types are independently distributed, (iii) preferences are quasi-linear, i.e. $\pi_i = v_i(x,t) - p$ where p is the price paid for bundle x, (iv) values are private, i.e. $v_i(x,t) = v_i(x,t_i)$ and (v) $v_i(x,t_i)$ is continuously differentiable in t_i.

Now compare inequality (4.6), which stems from core constraints, with equality (4.4), which results from incentive compatibility. Imposing individual rationality shows that the only way in which both sets of conditions can be satisfied is that $\pi(\underline{t}_i) = 0$ and

$$R \;=\; V_N - \sum_{i \in W}(V_N - V_{N\setminus i}) \;=\; R^V, \tag{4.7}$$

i.e. the revenue of a core-selecting auction has to be equal to the Vickrey revenue for *every* valuation profile (not just in expectation).

Since any core-selecting auction is efficient it also generates the same surplus as the Vickrey auction. As a result, the sum of bidders' profits has to equal the sum of bidders' Vickrey profits. In fact, it is easy to show something stronger, namely that in any core-selecting auction a bidder's profit is equal to her Vickrey profit. Recall that for efficient assignments we have

$$R + \sum_{j \in W} \pi_j \;=\; V_N,$$

which, together with (4.5), implies that

$$\pi_i \;\leq\; V_N - V_{N\setminus i} \;=\; \pi_i^V$$

But from (4.3) and $\pi_i(\underline{t}_i) \;=\; 0$ we conclude $\pi_i \;=\; \pi_i^V$, again for every valuation profile. Defining two mechanisms as equivalent when they produce the same surplus, revenue, and bidders' profits for all possible type profiles, we have:

Proposition 3. *Any core-selecting auction is equivalent to the Vickrey auction. Hence, if the Vickrey outcome is not in the core, there exists no individually-rational, incentive-compatible, core-selecting auction.*

When does the Vickrey auction result in a core outcome? Milgrom (2004, p. 312) shows that a necessary and sufficient condition is that goods are substitutes (see also Gul and Stacchetti, 1999).[9]

Corollary 2. *Core-selecting auctions exist if and only if goods are substitutes, in which case they are equivalent to the Vickrey auction.*

[9]Following Milgrom (2004, p. 312) it is assumed here that the set of valuation profiles contains the set of additive value functions.

To illustrate, consider a slight adaptation of the example of Section 3: two local bidders have valuations v_i that are uniformly distributed on [0,1] and the global bidder has a value V for the package that is uniformly distributed on [0,2] and values for the individual items equal to $\frac{1}{2}(1-\alpha)V$, where $-1 \leq \alpha \leq 1$. Note that the global bidder's valuations are sub-additive for α negative and super-additive for α positive. Consider valuation profiles for which $v_i \geq V$, which have strictly positive measure, so that the local bidders win and their Vickrey profits are $\pi_1^V = v_1 - \frac{1}{2}(1-\alpha)V$ and $\pi_2^V = v_2 - \frac{1}{2}(1-\alpha)V$. Since welfare is equal to $v_1 + v_2$, the seller's revenue is

$$R^V \; = \; (1-\alpha)V.$$

Now consider the blocking coalition formed by the seller and the global bidder, which results in the constraint

$$R^V \; \geq \; V$$

i.e. the Vickrey auction results in core allocations only if $\alpha \leq 0$, i.e. when goods are substitutes.

Finally, our results have implications for the implementability of competitive equilibrium outcomes. Since the Vickrey auction is the unique core-selecting auction, the only point in the core that is implementable is the Vickrey outcome.

Corollary 3. *When goods are substitutes, the competitive equilibrium outcome can be implemented if and only if it coincides with the Vickrey outcome. When goods are not substitutes, the competitive equilibrium outcome (if it exists) cannot be implemented.*

For the substitutes case, it is well known that Vickrey prices may be less than competitive equilibrium prices. To illustrate, suppose there are $2M$ items (with $M \geq 2$) and two bidders with valuation functions $v_i(m, t_i) = v(m)t_i$ for $i = 1, 2$, where the t_i are independently and uniformly distributed on [0,1] and $v(m)$ is a concave function for $0 \leq m \leq 2M$. Consider type profiles for which the types are "close," e.g. $t_1 = t(1+\epsilon)$ and $t_2 = t(1-\epsilon)$ with ϵ small. An efficient outcome dictates that both bidders get M units and the resulting Vickrey payments are approximately $(v(2M) - v(M))t$ for both bidders. In other words, the Vickrey per-unit price is

$$p^V \; = \; \frac{v(2M) - v(M)}{M} t.$$

A lower bound for the competitive equilibrium price, p, follows from the requirement that at price p neither bidder desires an additional unit: $v(M)t - Mp \geq v(M+1)t - (M+1)p$, or

$$p \geq \frac{v(M+1) - v(M)}{1} t,$$

which exceeds the Vickrey per-unit price by concavity of $v(\cdot)$. To summarize, for a positive measure of types, the competitive equilibrium outcome differs from the Vickrey outcome, and Corollary 3 implies that the competitive equilibrium outcome cannot be implemented.

5. Conclusion

The BCV auction has been shown to have some remarkable properties in complete-information environments where bidders' values and, hence, their bids are commonly known (e.g. Day and Milgrom, 2007). In particular, when bids are completely predictable, the introduction of BCV prices ensures outcomes that are "fair" and seller revenues that are not embarrassingly low.

This paper considers the performance of the BCV auction for the realistic case when bidders' values are privately known and, hence, their bids are not perfectly predictable. If in such incomplete-information environments truthful bidding would be optimal then the BCV auction would reliably outperform the Vickrey auction. However, our analysis shows that truthful bidding is *not* an equilibrium – bid shading is a "dominant property" in that it is optimal even when others bid truthfully. We show that the BCV auction may result in lower than Vickrey revenues and in inefficient outcomes that are on average *further* from the core than Vickrey outcomes.

We study whether core-allocations can be achieved by any mechanism in a general environment where goods can be substitutes and/or complements. We show that bidders' incentive compatibility constraints impose an upper bound on the seller's revenue. The upper bound is equal to the Vickrey revenue and has to hold for all possible valuation profiles (not just in expectation). As a result, total surplus, seller's revenue, and bidders' profits in any core-selecting auction are identical to their Vickrey counterparts, i.e. any core-selecting auction is equivalent to the Vickrey auction. A fortiori, if the Vickrey outcome is outside the core, no individually-rational, incentive-compatible core-selecting auction exists.

Our impossibility result is akin to Myerson and Satterthwaite's (1983) finding that efficient bilateral trade is not generally possible. In both cases the intuition is that incentive compatibility requires that market participants have information rents (reflecting their private information). In the two-sided setting studied by Myerson and Satterthwaite, traders' expected information rents may exceed the surplus generated by trade. In the one-sided setting studied here, bidders' information rents imply an upper-bound on the seller's revenue – an upper-bound that generally conflicts with some of the core constraints.

Core allocations are possible when all goods are substitutes. The interest in core-selecting auctions, however, derives from environments in which the substitutes assumption is relaxed. In this case, competitive equilibrium does not necessarily exist and "... the conception of auctions as mechanisms to identify market clearing prices is fundamentally misguided" (Milgrom, 2004, p. 296). The core, which always exist in these one-sided applications, seems the natural and relevant solution concept since "... competitive equilibrium outcomes are always core outcomes, so an outcome outside the core can be labeled uncompetitive" (Milgrom, 2004, p. 303). Our results, however, demonstrate that with incomplete information, core assignments are not generally possible unless all goods are substitutes. More generally, our results underline the importance of putting incentive constraints first, also in practical auction design.

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