ECON 501B 2016 MIDTERM EXAM
SOLUTIONS

① ASSUME THAT $(\hat{p}, \hat{x})$ IS A W.E.: IT SATISFIES

(U-MAX) $\hat{x}^i$ max's $u^i(\cdot)$ s.t. $\hat{p}\cdot x^i \leqq \hat{p}\cdot \mathring{x}^i$ & $x^i \geqq 0$ $(i=1,2)$

(M-CLR) $\hat{x}^1_k + \hat{x}^2_k \leqq \mathring{x}^1_k + \mathring{x}^2_k$ & $\hat{p}_k(\hat{x}^1_k + \hat{x}^2_k) = \hat{p}_k(\mathring{x}^1_k + \mathring{x}^2_k)$ $(k=1,\dots,\ell)$.

SUPPOSE $\hat{x}$ IS NOT A PARETO ALLOCATION:

WLOG SUPPOSE THAT $(\tilde{x}^1, \tilde{x}^2)$ SATISFIES

(a) $\tilde{x}^1_k + \tilde{x}^2_k \leqq \mathring{x}^1_k + \mathring{x}^2_k$ $(k=1,\dots,\ell)$

(b1) $u^1(\tilde{x}^1) > u^1(\hat{x}^1)$

(b2) $u^2(\tilde{x}^2) \geqq u^2(\hat{x}^2)$.

(b1) AND (U-MAX) YIELD $\hat{p}\cdot\tilde{x}^1 > \hat{p}\cdot\mathring{x}^1 \geqq \hat{p}\cdot\hat{x}^1$.

(b2) AND (U-MAX) YIELD $\hat{p}\cdot\tilde{x}^2 \geqq \hat{p}\cdot\hat{x}^2$, ACCORDING TO THE ALTERNATIVE DUALITY THEOREM.

WE THEREFORE HAVE $\hat{p}\cdot(\tilde{x}^1 + \tilde{x}^2) > \hat{p}\cdot(\hat{x}^1 + \hat{x}^2)$,

AND THEREFORE $\hat{p}_k(\tilde{x}^1_k + \tilde{x}^2_k) > \hat{p}_k(\hat{x}^1_k + \hat{x}^2_k)$ FOR SOME k,

WHICH YIELDS $\hat{p}_k > 0$ AND $\tilde{x}^1_k + \tilde{x}^2_k > \hat{x}^1_k + \hat{x}^2_k$.

SINCE $\hat{p}_k > 0$, (M-CLR) YIELDS $\hat{x}^1_k + \hat{x}^2_k = \mathring{x}^1_k + \mathring{x}^2_k$.

WE THEREFORE HAVE $\tilde{x}^1_k + \tilde{x}^2_k > \mathring{x}^1_k + \mathring{x}^2_k$,

WHICH CONTRADICTS (a). THEREFORE $\hat{x}$ IS A PARETO ALLOCATION. ||

(2) (a) FIRST NOTE THAT IF EITHER PRICE IS ZERO THERE WILL CLEARLY BE EXCESS DEMAND FOR THAT GOOD (IN FACT, NEITHER CONSUMER WOULD HAVE A UTILITY-MAXIMIZING BUNDLE). THEREFORE $p_x, p_y > 0$ ← IN A W.E.
LET $p_y \equiv 1$. IN A W.E. WE HAVE:

EACH FIRM j HAS $q_j = 2z_j$, AND $R_j = p_y q_j = q_j = 2z_j$ AND $C_j = p_x z_j$; $\therefore \pi_j = R_j - C_j = 2z_j - p_x z_j = (2-p_x)z_j$.
THEREFORE PROFIT-MAXIMIZATION REQUIRES THAT $p_x \geqq 2$ AND IF EITHER $z_j > 0$ THEN $p_x = 2$. LET'S ASSUME FIRST THAT SOME $z_j > 0$, SO WE HAVE $p_x = 2$.

EACH CONSUMER HAS $MRS^i = \frac{y_i}{x_i}$, SO UTILITY-MAXIMIZATION BY i REQUIRES $\frac{y_i}{x_i} = \frac{p_x}{p_y} = 2$; i.e., $y_i = 2x_i$, AND ALSO

$$p_x x_i + p_y y_i = p_x \mathring{x}_i + p_y \mathring{y}_i = 2\mathring{x}_i = 16,$$

$$\text{i.e., } 2x_i + y_i = 16; \text{ i.e., } 2x_i + 2x_i = 16. \therefore x_i = 4, y_i = 8.$$

THEREFORE $y_1 + y_2 = 16$, SO $q_1 + q_2 = 16$, AND $z_1 + z_2 = 8$.
WE HAVE $x_1 + x_2 + z_1 + z_2 = 4 + 4 + 8 = 16 = \mathring{x}$, SO MARKETS DO CLEAR.

WE ASSUMED THAT $z_1 > 0$ OR $z_2 > 0$. CAN THERE BE AN EQUILIBRIUM WITH $z_1 = z_2 = 0$? NO, BECAUSE THERE WOULD BE EXCESS DEMAND FOR Y.

SUMMARIZING: $(x_1, y_1) = (x_2, y_2) = (4, 8)$; $z_1 + z_2 = 8$;
$q_1 + q_2 = 16$; $q_1 = 2z_1$, $q_2 = 2z_2$. $p_x = 2$, $p_y = 1$.
$R_j = 2z_j$, $C_j = 2z_j$, $\pi_j = 0$ $(j = 1, 2)$.

(b) (P-MAX): $\max u^1(x_1,y_1) = x_1 y_1$ s.t. $x_i, y_i, z_j \geqq 0$ $(\forall i,j)$,

AND TO $x_1 + x_2 + z_1 + z_2 \leqq \mathring{x} = 16$

$$y_1 + y_2 \leqq q_1 + q_2 = 2z_1 + 2z_2$$

$$u^2(x_2,y_2) \geqq u_2, \text{ i.e., } x_2 y_2 \geqq u_2.$$

WE WANT TO SHOW THAT THE SOLUTION IN (a) SATISFIES THE FIRST-ORDER CONDITIONS FOR (P-MAX) — i.e., THAT THERE ARE VALUES OF u_2 AND OF $\sigma_x, \sigma_y, \lambda \geqq 0$ [WE'LL FIND $\underline{\sigma_x, \sigma_y, \lambda > 0}$] THAT SATISFY

SO FOC ARE EQUATIONS

x_1	(1)	$u^1_x = \sigma_x$, i.e., $y_1 = \sigma_x$	SO LET $\sigma_x = y_1 = 8$
y_1	(2)	$u^1_y = \sigma_y$, i.e., $x_1 = \sigma_y$	SO LET $\sigma_y = x_1 = 4$
x_2	(3)	$\lambda u^2_x = \sigma_x$, i.e. $\lambda y_2 = \sigma_x$	SO LET $\lambda = \sigma_x / y_2 = 1$
y_2	(4)	$\lambda u^2_y = \sigma_y$, i.e. $\lambda x_2 = \sigma_y$	i.e., $(1)(4) = 4$ OK
z_1	(5)	$0 = \sigma_x - 2\sigma_y$; i.e. $\sigma_x = 2\sigma_y$	i.e., $8 = (2)(4)$ OK
z_2	(6)	$0 = \sigma_x - 2\sigma_y$; i.e., $\sigma_x = 2\sigma_y$	i.e., $8 = (2)(4)$ OK
σ_x	(7)	$x_1 + x_2 + z_1 + z_2 = \mathring{x} = 16$	$4+4+4+4 = 16$ OK
σ_y	(8)	$y_1 + y_2 = 2(z_1 + z_2)$	$8+8 = 2(4+4)$ OK
λ	(9)	$u_2(x_2,y_2) = u_2$	SO LET $u_2 = (4)(8) = 32$.

THEREFORE ALL THE FIRST-ORDER CONDITIONS ARE SATISFIED AT THE SOLUTION IN (a) IF WE CHOOSE $u_2 = 32$, $\sigma_x = 8$, $\sigma_y = 4$, AND $\lambda = 1$.

THE SECOND-ORDER CONDITIONS ARE SATISFIED BECAUSE $u^1(\cdot)$ IS (STRICTLY) QUASICONCAVE, EACH CONSTRAINT FUNCTION IS QUASICONVEX, AND THE INTERIOR OF THE CONSTRAINT SET (THE FEASIBLE SET) IS NONEMPTY.

(3) $N=\{1,2,3\}$ $u^i(x,y)$, $i=1,2,3$. $MRS^i=\frac{y_i}{x_i}$, $i=1,2,3$.

$(\mathring{x}_1,\mathring{y}_1)=(\mathring{x}_2,\mathring{y}_2)=(12,0)$; $(\mathring{x}_3,\mathring{y}_3)=(0,12)$. $\Sigma(\mathring{x}_i,\mathring{y}_i)=(24,12)$.

$|S|=3$: $S=N$; PARETO REQUIRES $y_i=\frac{1}{2}x_i$ $(i=1,2,3)$ AND $\Sigma x_i=24$.

$|S|=1$: $S=\{1\},\{2\},\{3\}$; $\mathring{u}_i=0$ $(i=1,2,3)$.

$|S|=2$:

$S=\{1,2\}$: $(\mathring{x}_S,\mathring{y}_S)=(24,0)$, so S CAN DO NO BETTER THAN $u_1=u_2=0=\mathring{u}_1=\mathring{u}_2$.

$S=\{1,3\}$: $(\mathring{x}_S,\mathring{y}_S)=(12,12)$ AND THE UTILITY FRONTIER IS $\sqrt{u_1}+\sqrt{u_3}=\sqrt{\mathring{x}_S\mathring{y}_S}=12$.

ALTERNATIVELY: $((x_1,y_1),(x_3,y_3))$ IS PARETO FOR S IF $(x_1,y_1)+(x_3,y_3)=(12,12)$ AND $MRS^1=MRS^3$.

$S=\{2,3\}$: SAME AS FOR $S=\{1,3\}$.

(a) N CAN IMPROVE: (EQUAL-MRS) IS VIOLATED, SO THE ALLOCATION IS NOT PARETO OPTIMAL.

THE PROPOSAL YIELDS $u_1=u_2=16$ AND $u_3=64$, SO NO COALITION $S\neq N$ CAN IMPROVE ON IT:

$u_i>0=\mathring{u}_i$ $(\forall i)$, SO NO S WITH $|S|=1$ CAN IMPROVE.

$\sqrt{u_1}+\sqrt{u_2}=4+4=8>0$, SO $\{1,2\}$ CAN'T IMPROVE.

$\sqrt{u_1}+\sqrt{u_3}=\sqrt{u_2}+\sqrt{u_3}=4+8=12=\sqrt{\mathring{x}_S\mathring{y}_S}$, SO $\{1,3\}$ AND $\{2,3\}$ CAN'T IMPROVE.

~~A PARETO IMPROVEMENT: $(x_1,y_1)=(x_2,y_2)=(6,3)$; $(x_3,y_3)=(12,6)$, WHICH IS IN FACT PARETO OPTIMAL TOO.~~

(b) THE PROPOSAL IS PARETO OPTIMAL, BUT $\{1,3\}$ AND $\{2,3\}$ CAN BOTH IMPROVE ON IT (NOT SIMULTANEOUSLY, OF COURSE!):

FOR EXAMPLE, $(x_1,y_1)=(x_3,y_3)=(6,6)$; $u_1=u_3=36>32$.

(c) THIS PROPOSAL IS ~~NOT~~ IN THE CORE:

IT'S PARETO OPTIMAL (N CAN'T IMPROVE ON IT):

$MRS^i=\frac{1}{2}$ $(\forall i)$ AND $\sum(x_i,y_i)=(24,12)=(\mathring{x},\mathring{y})$.

FOR $S=\{1,3\}$:

$\sqrt{u_1}+\sqrt{u_3}=\sqrt{8}+\sqrt{98}=2\sqrt{2}+7\sqrt{2}=9\sqrt{2}>12=\sqrt{\mathring{x}_S\mathring{y}_S}$,

SO $\{1,3\}$ CAN'T IMPROVE. SIMILARLY FOR $S=\{2,3\}$.

FOR $S=\{1,2\}$: $\sqrt{u_1}+\sqrt{u_2}=\sqrt{8}+\sqrt{18}>0=\sqrt{\mathring{x}_S\mathring{y}_S}$,

SO $\{1,2\}$ CAN'T IMPROVE.

FOR $|S|=1$: $u_i>0=\mathring{u}_i$ FOR $i=1,2,3$, SO NO 1-PERSON COALITION CAN IMPROVE.

~~[illegible]~~