ECON 501B FALL 2009
MIDTERM EXAM
SOLUTIONS

① (a) $\max\limits_{x_A, y_A, x_B, y_B} \quad 2\alpha\sqrt{x_A} + 2\sqrt{y_A}$ s.t. $x_A, y_A, x_B, y_B \geqq 0$

AND $x_A + x_B \leqq \mathring{x}$ $\qquad \sigma_x$

$y_A + y_B \leqq \mathring{y}$ $\qquad \sigma_y$

$2\alpha\sqrt{x_B} + 2\sqrt{y_B} \geqq \bar{u}_B$ $\qquad \lambda_B$

INTERIOR FOC: $\exists\, \sigma_x, \sigma_y, \lambda_B \geqq 0$ s.t.

$$\left.\begin{array}{l} x_A: \dfrac{\alpha}{\sqrt{x_A}} = \sigma_x \\ y_A: \dfrac{1}{\sqrt{y_A}} = \sigma_y \end{array}\right\} \alpha\sqrt{\frac{y_A}{x_A}} = \frac{\sigma_x}{\sigma_y}$$

$$\left.\begin{array}{l} x_B: \lambda_B\dfrac{\alpha}{\sqrt{x_B}} = \sigma_x \\ y_B: \lambda_B\dfrac{1}{\sqrt{y_B}} = \sigma_y \end{array}\right\} \alpha\sqrt{\frac{y_B}{x_B}} = \frac{\sigma_x}{\sigma_y}$$

$$\therefore \frac{y_A}{x_A} = \frac{y_B}{x_B}$$

AND SINCE THE THREE LAGRANGE MULTIPLIERS ARE CLEARLY POSITIVE, ALL THREE CONSTRAINTS MUST BE SATISFIED AS EQUATIONS.

LET (x_A, y_A, x_B, y_B) BE ANY PARETO ALLOCATION.
WE HAVE $\dfrac{y_A}{x_A} = \dfrac{y_B}{x_B} = r$, LET'S SAY. THEN

$y_A = rx_A$ AND $y_B = rx_B$ AND

$$\mathring{y} = y_A + y_B = rx_A + rx_B = r(x_A + x_B) = r\mathring{x};$$

i.e., $r = \dfrac{\mathring{y}}{\mathring{x}}$, AND $\dfrac{y_A}{x_A} = \dfrac{y_B}{x_B} = \dfrac{\mathring{y}}{\mathring{x}} = r$ AT EVERY PARETO ALLOCATION.

$\therefore$ EFFICIENCY PRICES SATISFY

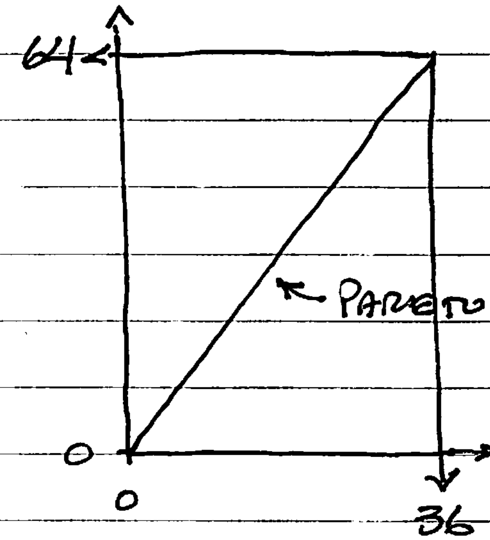
$$\frac{\sigma_x}{\sigma_y} = \alpha\sqrt{\frac{y_i}{x_i}} = \alpha\sqrt{r} = \alpha\sqrt{\frac{\mathring{y}}{\mathring{x}}}$$ AT EVERY PARETO ALLOCATION.

(b) $r = \frac{\mathring{y}}{\mathring{x}} = \frac{64}{36} = \frac{16}{9}$ AND

$y_A = \frac{16}{9} x_A$ AND $y_B = \frac{16}{9} x_B$.

THE PARETO ALLOCATIONS ARE THE ONES THAT SATISFY

$$x_A + x_B = 36$$
$$y_A + y_B = 64$$
$$y_A = \frac{16}{9} x_A, \; y_B = \frac{16}{9} x_B.$$



(c) WE STILL HAVE $(\mathring{x}, \mathring{y}) = (36, 64)$, SO AT EVERY PARETO ALLOCATION WE HAVE THE SAME EFFICIENCY PRICES:

$$\frac{\sigma_x}{\sigma_y} = \sqrt{\frac{\mathring{y}}{\mathring{x}}} = \sqrt{\frac{16}{9}} = \frac{4}{3}.$$

THE FIRST WELFARE THEOREM ENSURES THAT A WALRASIAN ALLOCATION MUST BE ONE OF THE PARETO ALLOCATIONS, AND THE WALRASIAN PRICES WILL BE THE ONES FOR THAT ALLOCATION. SINCE THE PARETO ALLOCATIONS ALL HAVE THE SAME EFFICIENCY PRICES, $\frac{\sigma_x}{\sigma_y} = \frac{4}{3}$, THE EQUILIBRIUM PRICES MUST SATISFY $\frac{p_x}{p_y} = \frac{4}{3}$.

LET $p_x = 4$ AND $p_y = 3$. THEN

$W_A = (4)(21) + (3)(6) = 84$ AND $y_A = \frac{16}{9} x_A$;

i.e., $4x_A + 3\left(\frac{16}{9} x_A\right) = 84 \Rightarrow 4x_A + \frac{48}{9} x_A = 84$;

$\frac{36}{9} x_A + \frac{48}{9} x_A = 84$; $\frac{84}{9} x_A = 84$; $x_A = 9, y_A = 16$

$\therefore x_B = 27, y_B = 48$.

CHECK:

$MRS_A = \frac{4}{3} = \frac{p_x}{p_y}$ AND $MRS_B = \frac{4}{3} = \frac{p_x}{p_y}$.

② (a) ASSUME THAT $z_k(p) \leqq 0$ FOR EACH k. THEN WE ONLY NEED TO SHOW THAT $p_k z_k(p) = 0$ FOR EACH k. SINCE $p_k \geqq 0, \forall k$, WE HAVE $p_k z_k(p) \leqq 0, \forall k$. WALRAS' LAW IS THAT $\sum p_k z_k(p) = 0$. SINCE NONE OF THE TERMS IS POSITIVE, ALL ARE ZERO: $p_k z_k(p) = 0, \forall k$.

(b) ASSUME THAT p IS A FIXED POINT OF f; THEN

$$\frac{1}{\sum [p_k + M_k(p)]} [p + M(p)] = p$$

i.e., $p + M(p) = \sum [p_k + M_k(p)]\, p$

$$\therefore \quad p \cdot z(p) + M(p) \cdot z(p) = \sum [p_k + M_k(p)]\, p \cdot z(p)$$

WALRAS' LAW YIELDS $p \cdot z(p) = 0$, SO WE HAVE $M(p) \cdot z(p) = 0$, i.e., $\sum M_k(p) z_k(p) = 0$.

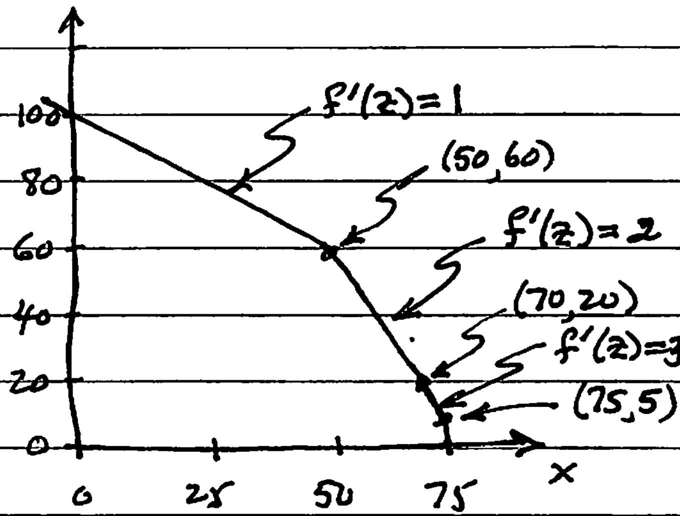
FOR EACH k, THE DEFINITION OF M_k YIELDS $M_k(p) \geqq 0$ AND ALSO $M_k(p) z_k(p) \geqq 0$. COMBINED WITH $\sum M_k(p) z_k(p) = 0$, THIS YIELDS $M_k(p) z_k(p) = 0, \forall k$. THIS IN TURN YIELDS $z_k(p) \leqq 0, \forall k$, SINCE $z_k(p) > 0 \Rightarrow M_k(p) > 0$.

(c) S IS NONEMPTY, COMPACT, AND CONVEX. SINCE f IS A SUM AND QUOTIENT OF CONTINUOUS FUNCTIONS, AND MAX AND THE DENOMINATOR FUNCTION IS NEVER ZERO, f ITSELF IS A CONTINUOUS FUNCTION. BROUWER'S THEOREM THEREFORE ENSURES THAT f HAS A FIXED POINT.

③ (a)

$$q = f(z) = \begin{cases} 3z, & z \leqq 5 \\ 5 + 2z, & 5 \leqq z \leqq 25 \\ 30 + z, & z \geqq 25 \end{cases}$$

AND $(\overset{\circ}{x}, \overset{\circ}{y}) = (75, 5)$.



(b) $f'(z) = MRS_A = MRS_B$,

IN EACH CASE THE MARGINAL VALUE OF THE X-GOOD MEASURED IN TERMS OF THE Y-GOOD.

(c)(c1) $(x_A, y_A) = (x_B, y_B) = (35, 10)$:

$MRS_i = 3\left(\frac{10}{35}\right) = \frac{30}{35} = \frac{6}{7}$ AND $z = 5$, SO $f'(\tilde{z}) = 2$ FOR LARGER VALUES OF $\tilde{z}$. WE CAN THEREFORE MAKE A PARETO IMPROVEMENT BY SHIFTING SOME x TO z AND INCREASING q AND y:

SAY, $\Delta z = +2$, $\Delta y = \Delta q = +4$; $\Delta x_i = -1$, $\Delta y_i = +2$, $\forall i$.

WE GET $(\tilde{x}_A, \tilde{y}_A) = (\tilde{x}_B, \tilde{y}_B) = (34, 12)$, WHICH MAKES EACH CONSUMER BETTER OFF, SINCE EVEN HERE WE STILL HAVE $MRS_i = \frac{36}{34} < 2 = f'(\tilde{z})$.

(c2) (48,21) AND (16,7): $z = 11$, $q = 23$

$MRS_A = MRS_B = \frac{21}{16} < 2 = f'(z)$, SO JUST AS IN (c1), WE CAN DO

$$\Delta z = +2,\ \Delta y = \Delta q = +4,\ \Delta x_i = -1,\ \Delta y_i = +2,\ i = A, B.$$

WE GET $(\tilde{x}_A, \tilde{y}_A) = (47, 23)$ AND $(\tilde{x}_B, \tilde{y}_B) = (15, 9)$.

(c3) (45, 30) AND (15, 10):

IF $z = 15$ AND $q = f(z) = 35$, WE HAVE

$$x_A + x_B + z = 45 + 15 + 15 = 75 = \mathring{x}$$

$$y_A + y_B = 30 + 10 = 40 = \mathring{y} + q$$

SO THIS ALLOCATION IS FEASIBLE AND NON-WASTEFUL.

ALSO, $MRS_A = MRS_B = 2 = f'(z)$, SO IT IS PARETO OPTIMAL.

THE EFFICIENCY PRICES SATISFY

$$\frac{\sigma_x}{\sigma_y} = f'(z) = MRS_A = MRS_B = 2.$$

(d) WITH $p_x = \$2$ AND $p_y = \$1$, THESE ARE THE EFFICIENCY PRICES IN (c3), AND $q = 35$ IS THE PRODUCTION LEVEL THERE, SO LET'S CONSIDER THAT ALLOCATION:

$$z = 15, \quad q = f(z) = 35; \quad f'(z) = 2.$$

$$(x_A, y_A) = (45, 30) \text{ AND } (x_B, y_B) = (15, 10).$$

PROFIT: $\pi = (\$1)(35) - (\$2)(15) = \$35 - \$30 = \$5.$

$W_A = (\$2)(55) + (\$1)(5) + \$5 = \$110 + \$5 + \$5 = \$120.$

$p_x x_A + p_y y_A = (2)(45) + (1)(30) = \$120 = W_A$

AND $MRS_A = 2 = \dfrac{p_x}{p_y}$, SO A IS MAXIMIZING u_A.

$W_B = (\$2)(20) + (\$1)(0) = \$20.$

$p_x x_B + p_y y_B = (2)(15) + (1)(10) = \$40 = W_B$

AND $MRS_B = 2 = \dfrac{p_x}{p_y}$, SO B IS MAXIMIZING u_B.

$f'(z) = 2 = \dfrac{p_x}{p_y}$ AND $f''(z) = 0$, SO THE FIRM IS MAXIMIZING PROFIT.

MARKET CLEARANCE WAS ESTABLISHED IN (c3).