ECON 501B FALL 2009 FINAL EXAM
SOLUTIONS

(1) $MRS_H^A = \dfrac{20}{x_H^A}$, $MRS_F^A = \dfrac{5}{x_F^A}$ $\quad (\mathring{x}_0^A, \mathring{x}_H^A, \mathring{x}_F^A) = (40, 50, 30)$

$MRS_H^B = \dfrac{10}{x_H^B}$, $MRS_F^B = \dfrac{10}{x_F^B}$ $\quad (\mathring{x}_0^B, \mathring{x}_H^B, \mathring{x}_F^B) = \underline{(20, 10, 15)}$

$60, 60, 45$

(a) PARETO: $\dfrac{20}{x_H^A} = \dfrac{10}{x_H^B}$; $x_H^A = 2x_H^B$; $x_H^A = 40$, $x_H^B = 20$

$MRS_H^A = MRS_H^B = \frac{1}{2}$

$\dfrac{5}{x_F^A} = \dfrac{10}{x_F^B}$; $x_F^B = 2x_F^A$; $x_F^A = 15$, $x_F^B = 30$

$MRS_F^A = MRS_F^B = \frac{1}{3}$

(b) ARROW-DEBREU: $p_0 = 1$, $p_H = \frac{1}{2}$, $p_F = \frac{1}{3}$

$x_H^A, x_F^A, x_H^B, x_F^B$ AS ABOVE

$x_0^A = \mathring{x}_0^A - \frac{1}{2}(40-50) - \frac{1}{3}(15-30) = 40+5+5 = 50$

$x_0^B = \mathring{x}_0^B - \frac{1}{2}(20-10) - \frac{1}{3}(30-15) = 20-5-5 = 10$

(c) $\dfrac{1}{1+r} = p_H + p_F = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$; $\therefore 1+r = \frac{6}{5}$ AND $r = \frac{1}{5} = 20\%$.

(d) $d_\gamma = \begin{bmatrix}1\\2\end{bmatrix}$, $d_\delta = \begin{bmatrix}1\\0\end{bmatrix}$; $\therefore q_\gamma = (1)(\frac{1}{2}) + (2)(\frac{1}{3}) = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

$q_\delta = (1)(\frac{1}{2}) + (0)(\frac{1}{3}) = \frac{1}{2}$

(e) $y = (\frac{1}{2}, \frac{1}{2})$ YIELDS $\frac{1}{2}d_\gamma + \frac{1}{2}d_\delta = \begin{bmatrix}1/2\\1\end{bmatrix} + \begin{bmatrix}1/2\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

COST OF y: $\frac{1}{2}(\frac{7}{6}) + \frac{1}{2}(\frac{1}{2}) = \frac{7}{12} + \frac{3}{12} = \frac{10}{12} = \frac{5}{6}$

$\therefore \dfrac{1}{1+r} = \frac{5}{6}$ AND $r = \frac{1}{5} = 20\%$ AS ABOVE.

$\begin{bmatrix}x_H^A - \mathring{x}_H^A\\ x_F^A - \mathring{x}_F^A\end{bmatrix} = \begin{bmatrix}-10\\-15\end{bmatrix} = y_\gamma^A\begin{bmatrix}1\\2\end{bmatrix} + y_\delta^A\begin{bmatrix}1\\0\end{bmatrix}$; $\therefore y_\gamma^A = -7\frac{1}{2}$, $y_\delta^A = -2\frac{1}{2}$

SIMILARLY, $y_\gamma^B = 7\frac{1}{2}$, $y_\delta^B = 2\frac{1}{2}$

② $u_A = y_A + 8x - \frac{1}{2}x^2$, $MRS_A = 8 - x$, IF $x \leqq 8$

$u_B = y_B + 12x - \frac{1}{2}x^2$, $MRS_B = 12 - x$, IF $x \leqq 12$

$C(x) = 4x$, $MC = 4$

(a) $MRS_A = MC$: $8 - x = 4$, $\therefore x = 4$.

(b) LET $\overset{\circ}{u}_A = \overset{\circ}{y}_A$ AND $\overset{\circ}{u}_B = \overset{\circ}{y}_B$ (i.e., UTILITIES IF $x = 0$).

AT $x = 4$:

$$u_A = \overset{\circ}{y}_A \underbrace{- 16}_{C(4)} + (8)(4) - \tfrac{1}{2}(4)^2 = \overset{\circ}{y}_A - 16 + 32 - 8 = \overset{\circ}{y}_A + 8$$

$$u_B = \overset{\circ}{y}_B + (12)(4) - \tfrac{1}{2}(4)^2 = \overset{\circ}{y}_B + 48 - 8 = \overset{\circ}{y}_B + 40$$

$\therefore$ $CS_A = 8$, $CS_B = 40$ AND $CS = 48$ IN TOTAL.

(c) MAX $\lambda_A u_A(x, y_A)$ s.t. $x, y_A, y_B \geqq 0$ AND $y_A + y_B + C(x) \leqq \overset{\circ}{y}$ AND $u_B(x, y_B) \geqq \overline{u}_B$.

<u>FOMC (INTERIOR):</u>

x: $\lambda_A u_{Ax} + \lambda_B u_{Bx} = \sigma C'(x)$

y_A: $\lambda_A u_{Ay} = \sigma$, i.e., $\lambda_A = \frac{\sigma}{u_{Ay}}$

y_B: $0 = \sigma - \lambda_B u_{By}$, i.e., $\lambda_B = \frac{\sigma}{u^B_y}$

COMBINING: $\sigma \frac{u_{Ax}}{u_{Ay}} + \sigma \frac{u_{Bx}}{u_{By}} = \sigma C'(x)$

i.e., $MRS_A + MRS_B = MC$.

(d) FOR ALICE AND BART: $(8 - x) + (12 - x) = 4$ IF $x \leqq 8$

i.e., $20 - 2x = 4$; $2x = 16$; $\boxed{\hat{x} = 8}$

$$\hat{u}_A = \overset{\circ}{y}_A \underbrace{- 32}_{C(8)} + (8)(8) - \tfrac{1}{2}(8)^2 = \overset{\circ}{y}_A - 32 + 64 - 32 = \overset{\circ}{y}_A \qquad CS_A = 0$$

$$\hat{u}_B = \overset{\circ}{y}_B + (12)(8) - \tfrac{1}{2}(8)^2 = \overset{\circ}{y}_B + 96 - 32 = \overset{\circ}{y}_B + 64 \qquad CS_B = 64$$

$CS = 64$.

(e) IF B PAYS t TO A TO PRODUCE AT $x = 8$:

$$u_A = \hat{u}_A + t = \mathring{y}_A + t \qquad u_A \geq \mathring{u}_A \Rightarrow t \geq 8$$

$$u_B = \hat{u}_B - t = \mathring{y}_B + 64 - t \qquad u_B \geq \mathring{u}_B \Rightarrow t \leq 24.$$

(f) IF A CAN CHOOSE TO EXCLUDE B OR NOT, AND CHARGE B A PRICE p THAT MAXIMIZES A'S PROFIT:

~~B'S DEMAND IS $x = 12 - p$, i.e., $p = 12 - x$; $MR = 12 - 2x$.~~

~~$MR = MC$: $12 - 2x = 4$; $\therefore 2x = 8$; $x = 4$. $p = \$8$.~~

$\therefore R = (\$8)(4)$, $C = (\$4)(4)$, $\pi = R - C = \$16$.

$$u_A = \mathring{y}_A + \overset{\pi}{16} + 32 - 8 = \mathring{y}_A + 40 \qquad CS_A + \pi = 40$$

$$u_B = \mathring{y}_B - \underset{px}{32} + 48 - 8 = \mathring{y}_B + 8 \qquad CS_B = 8$$

TOTAL SURPLUS $= 48$

(g) A SMALL CHANGE IN p AND x WILL LEAVE π VIRTUALLY UNCHANGED, SINCE THE CURRENT p AND x MAXIMIZE π. BUT $MRS_A = 4$ AT $x = 4$, SO AN INCREASE IN x WILL INCREASE u_A. A DOES NOT NEED TO COMPARE MRS_A TO MC BECAUSE COST (AND MC) IS ALREADY ACCOUNTED FOR IN THE π-CALCULATION.

FOR SMALL CHANGES IN X

SEE THE FOLLOWING PAGE FOR THE PRODUCTION LEVEL x AND PRICE p THAT MAXIMIZE ALICE'S UTILITY. IT'S QUITE STRAIGHTFORWARD, BUT BECAUSE x AND p ARE NOT INTEGERS IT'S A BIT MESSY TO CALCULATE REVENUE, PROFIT, SURPLUS, ETC.

↙ THIS WAS NOT ASKED ON THE EXAM

If Alice chooses his production level x and the price p he will charge Bart so as to maximize his utility, he solves the problem

$$\max_{x,q} u_A(x, \mathring{y}_A - C(x) + R(q)), \text{ where } x = q$$

and $R(q) = pq = (12-q)q$.

[We'll check later whether he can do better by choosing $q < x$.]

FOMC:

$$\frac{\partial u_A}{\partial x} + \frac{\partial u_A}{\partial y} \cdot \frac{\partial y}{\partial R} \cdot \underbrace{\frac{\partial R}{\partial q}}_{=\frac{\partial R}{\partial x}} - \frac{\partial C}{\partial x} = 0$$

i.e. $MRS_A + (1)(1)MR = MC$

i.e., $(8-x) + (12-2x) = 4$

i.e., $20 - 3x = 4$; i.e., $3x = 16$

i.e., $\boxed{x = \frac{16}{3} = 5\tfrac{1}{3}}$

Since $q = x$ and $p = 12 - q$, we have $\boxed{p = \$6\tfrac{2}{3}}$

and revenue is $R(5\tfrac{1}{3}) = (\$6\tfrac{2}{3})(5\tfrac{1}{3}) = (\frac{20}{3})\frac{16}{3} = \frac{320}{9} = \$35\tfrac{5}{9}$

Cost is $C(5\tfrac{1}{3}) = (\$4)(5\tfrac{1}{3}) = \frac{64}{3} = \$21\tfrac{1}{3}$.

$u_A = \mathring{y}_A - 21\tfrac{1}{3} + 35\tfrac{5}{9} + (8)(5\tfrac{1}{3}) - \frac{1}{2}(5\tfrac{1}{3})^2$

$= \mathring{y}_A + 14\tfrac{2}{9} + 42\tfrac{2}{3} - \frac{1}{2}(\frac{256}{9}) = \mathring{y}_A + 42\tfrac{2}{3} + 14\tfrac{2}{9} - 14\tfrac{2}{9}$

$= \mathring{y}_A + 42\tfrac{2}{3}$

$u_B = \mathring{y}_B - 35\tfrac{5}{9} + (12)(5\tfrac{1}{3}) - \frac{1}{2}(\frac{256}{9}) = \mathring{y}_A + 64 - 35\tfrac{5}{9} - 14\tfrac{2}{9}$

$= \mathring{y}_B + 14\tfrac{2}{9}$. $\quad CS = CS_A + CS_B = 42\tfrac{2}{3} + 14\tfrac{2}{9} = 56\tfrac{8}{9}$.

SUMMARIZING:

$X = q = 5\frac{1}{3}$, $p = \$6\frac{2}{3}$, $R = \$35\frac{5}{9}$, $C = \$21\frac{1}{3}$, $\pi = \$14\frac{2}{9}$

$u_A = \mathring{y}_A + 42\frac{2}{3}$, $CS_A = 42\frac{2}{3}$

$u_B = \mathring{y}_B + 14\frac{2}{9}$, $CS_B = 14\frac{2}{9}$

TOTAL SURPLUS $= 56\frac{8}{9}$.

NOW WE ASK "WOULD u_A BE LARGER IF $q < X$" — i.e., BY INCREASING X AND/OR DECREASING q (AND INCREASING p)?

WE HAVE $MRS_A + MR = MC$ AND BOTH MRS_A AND MR ARE POSITIVE. INCREASING X WITHOUT INCREASING q WILL DECREASE u_A, BECAUSE $MRS_A < MC$. CONVERSELY, DECREASING q WILL DECREASE REVENUE (BECAUSE $MR > 0$) BUT WILL NOT DECREASE COST (BECAUSE X IS NOT REDUCED), SO THIS WILL ALSO DECREASE u_A.

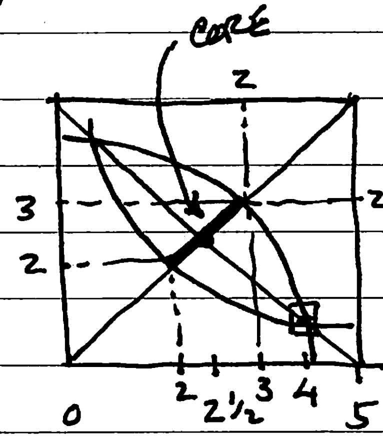
③ $u(x,y) = xy$ FOR EVERYONE; $(\overset{\circ}{x}_A, \overset{\circ}{y}_A) = (4,1)$, $(\overset{\circ}{x}_B, \overset{\circ}{y}_B) = (1,4)$.
$\overset{\circ}{u}_A = \overset{\circ}{u}_B = 4$

(a) WALRASIAN EQUIL'M: $p_x = p_y$; $(x_A, y_A) = (x_B, y_B) = (2\tfrac{1}{2}, 2\tfrac{1}{2})$.

VERIFY: $p_x x_i + p_y y_i = 2\tfrac{1}{2} + 2\tfrac{1}{2} = 5 = p_x \overset{\circ}{x}_i + p_y \overset{\circ}{y}_i$, $i = 1,2$

ALSO $x_A + x_B = \overset{\circ}{x}$
$y_A + y_B = \overset{\circ}{y}$ → $MRS_i = \dfrac{2\tfrac{1}{2}}{2\tfrac{1}{2}} = 1 = \dfrac{p_x}{p_y}$, $i = 1,2$.

(b) CORE:



PARETO REQUIRES $y_A = x_A$ AND $y_B = x_B$.

$\therefore\ u_A = x_A^2$ AND $u_B = x_B^2$.

INDIVIDUAL RATIONALITY REQUIRES

$u_A \geqq \overset{\circ}{u}_A$ AND $u_B \geqq \overset{\circ}{u}_B$,

i.e., $x_A^2 \geqq 4$ AND $x_B^2 \geqq 4$,

i.e., $x_A \geqq 2$ AND $x_B \geqq 2$.

(c) ADD $(\overset{\circ}{x}_c, \overset{\circ}{y}_c) = (2,2)$; $\overset{\circ}{u}_c = 4$.

WALRAS: STILL $p_x = p_y$, $(x_A, y_A) = (x_B, y_B) = (2\tfrac{1}{2}, 2\tfrac{1}{2})$;
ALSO $(x_c, y_c) = (2,2) = (\overset{\circ}{x}_c, \overset{\circ}{y}_c)$.

NOW WE HAVE $(x_A, y_A) + (x_B, y_B) + (x_c, y_c) =$
$= (2\tfrac{1}{2}, 2\tfrac{1}{2}) + (2\tfrac{1}{2}, 2\tfrac{1}{2}) + (2,2) = (7,7) = (\overset{\circ}{x}, \overset{\circ}{y})$.

CATHY IS ALSO MAXIMIZING HER UTILITY:
$(x_c, y_c) = (\overset{\circ}{x}_c, \overset{\circ}{y}_c)$ AND $MRS_c = 1 = \dfrac{p_x}{p_y}$.

(d) CORE:

PARETO AGAIN REQUIRES $x_i = y_i$ FOR EACH i, SO LET AN ALLOCATION BE REPRESENTED BY JUST (x_A, x_B, x_c), AND DEPICT ALLOCATIONS IN THE SIMPLEX $x_A + x_B + x_c = \overset{\circ}{x}_A + \overset{\circ}{x}_B + \overset{\circ}{x}_c = 7$. THIS TAKES CARE OF THE COALITION $S = \{A, B, C\}$.

ONE-PERSON COALITIONS $S=\{A\}, \{B\}$, OR $\{C\}$:
(INDIVIDUAL RATIONALITY)

$$u_i \geqq \mathring{u}_i = 4 \quad (i=A,B,C);\ \therefore\ x_i^2 \geqq 4,\ \text{i.e., } x_i \geqq 2 \quad (i=A,B,C).$$

TWO-PERSON COALITIONS:

$S=\{A,B\}$: $(\mathring{x}_S, \mathring{y}_S) = (5,5)$.

EFFICIENCY FOR S WITH $(\mathring{x}_S, \mathring{y}_S)$ REQUIRES

$$x_A = y_A,\ x_B = y_B, \text{ AND } x_A + x_B = \mathring{x}_S = 5.$$

$$\therefore u_A = x_A^2 \text{ AND } u_B = x_B^2;$$

$$\text{i.e., } \sqrt{u_A} = x_A \text{ AND } \sqrt{u_B} = x_B;\ \therefore \sqrt{u_A} + \sqrt{u_B} = \mathring{x}_S = 5.$$

SINCE S CAN ACHIEVE $\sqrt{u_A}+\sqrt{u_B}=5$, CORE ALLOCATION HAVE TO SATISFY $\sqrt{u_A}+\sqrt{u_B} \geqq 5$. SINCE CORE ALLOCATIONS ALSO SATISFY $u_A = x_A^2$ AND $u_B = x_B^2$, THIS YIELDS $x_A + x_B \geqq 5$, WHICH IN TURN YIELDS $x_C \leqq 2$.

$S=\{B,C\}$: $(\mathring{x}_S, \mathring{y}_S) = (3,6)$.

AS ABOVE, S-EFFICIENCY REQUIRES $y_B = 2x_B$, $y_C = 2x_C$, AND $x_B + x_C = \mathring{x}_S = 3$.

$$\therefore u_B = 2x_B^2 \text{ AND } u_C = 2x_C^2;$$

$$\text{i.e., } \sqrt{u_B} = \sqrt{2}\,x_B \text{ AND } \sqrt{u_C} = \sqrt{2}\,x_C;$$

$$\therefore \sqrt{u_B} + \sqrt{u_C} = \sqrt{2}\,(x_B + x_C) = 3\sqrt{2}.$$

CORE ALLOCATIONS (IN WHICH $\sqrt{u_B} = x_B$ AND $\sqrt{u_C} = x_C$) THEREFORE HAVE TO SATISFY $x_B + x_C \geqq 3\sqrt{2} \approx 4.242$; THIS IN TURN YIELDS $x_A \leqq 7 - 3\sqrt{2} \approx 2.758$.

$S=\{A,C\}$: $(\mathring{x}_S, \mathring{y}_S) = (6,3)$, AND A SYMMETRIC ARGUMENT TO $S=\{B,C\}$ YIELDS $x_A + x_C \geqq 3\sqrt{2} \approx 4.242$ AND $x_B \leqq 7 - 3\sqrt{2} \approx 2.758$.

SUMMARIZING, CORE ALLOCATIONS HAVE TO SATISFY THE FOLLOWING INEQUALITIES:

(1) $y_i = x_i$ $(i = A, B, C)$ AND $x_A + x_B + x_C = y_A + y_B + y_C = 7$.

(2) $x_i \geqq 2$ $(i = A, B, C)$

(3) $x_A + x_B \geqq 5$, $x_C \leqq 2$

(2) & (3) TOGETHER YIELD $x_C = 2$ AND $x_A + x_B = 5$

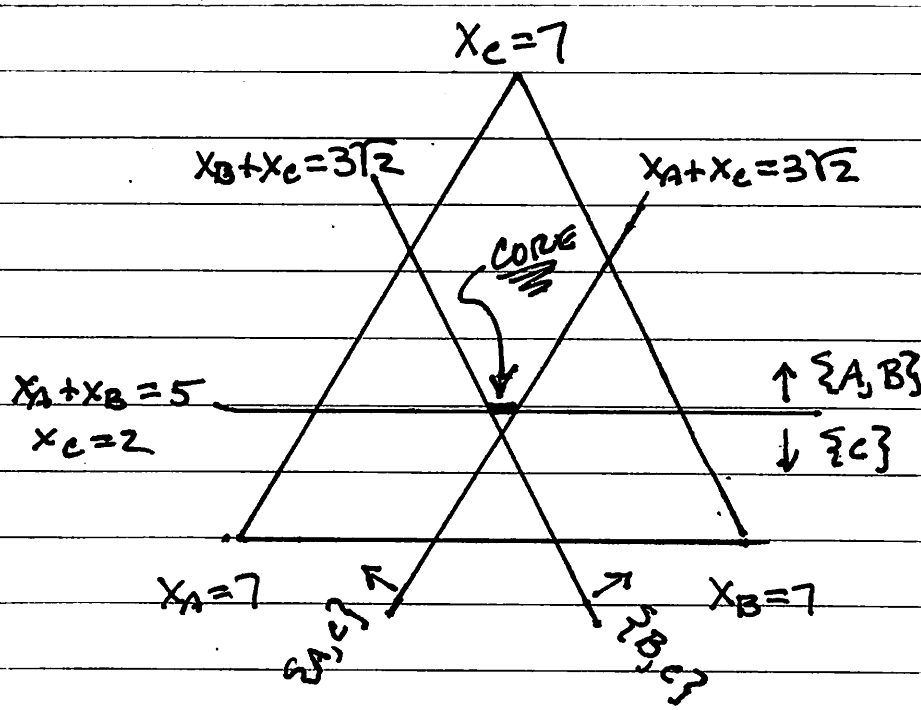
(4) $x_A + x_C \geqq 3\sqrt{2} \approx 4.242$, $x_B \leqq 7 - 3\sqrt{2} \approx 2.758$

(5) $x_B + x_C \geqq 3\sqrt{2} \approx 4.242$, $x_A \leqq 7 - 3\sqrt{2} \approx 2.758$

THESE CAN BE REDUCED TO

$x_i = y_i$ $(i = A, B, C)$, $x_A + x_B + x_C = 7$

$x_C = 2$, $x_A + x_B = 5$, $3\sqrt{2} \leqq x_A, x_B \leqq 7 - 3\sqrt{2}$.



THE INEQUALITIES FOR {A} AND {B} ARE NOT BINDING AND ARE NOT SHOWN.