ECON 501B FALL 2008

MIDTERM EXAM

SOLUTIONS

(1) (a) $\max_{x_A, y_A, x_B, y_B} \; x_A y_A$ s.t. $x_A, y_A, x_B, y_B \geqq 0$

AND TO

$$x_A + x_B \leqq 4 \; : \quad \sigma_x$$

$$y_A + y_B \leqq 8 \; : \quad \sigma_y$$

$$\underbrace{y_B - \tfrac{1}{8}(4 - x_B)^2}_{= u_B(\hat{x}_B, \hat{y}_B)} \geqq 0 \; : \quad \lambda$$

<u>FOC:</u> (NOTE THAT $x_A, y_A, y_B > 0$)

x_A: $\quad y_A = \sigma_x$ $\quad$ i.e., $\boxed{\sigma_x = 6}$

y_A: $\quad x_A = \sigma_y$ $\quad$ i.e., $\boxed{\sigma_y = 4}$

x_B: $\quad 0 \leqq \sigma_x - \lambda(1 - \frac{1}{4}x_B)$ $\quad$ i.e., $\lambda \leqq \sigma_x$ ok

y_B: $\quad 0 = \sigma_y - \lambda(1)$ $\quad$ i.e., $\sigma_y = \lambda$, i.e., $\boxed{\lambda = 4}$

σ_x: $\quad x_A + x_B = 4$ (SINCE $\sigma_x > 0$) $\quad$ i.e., $4 + 0 = 4$ ok

σ_y: $\quad y_A + y_B = 8$ (SINCE $\sigma_y > 0$) $\quad$ i.e., $6 + 2 = 8$ ok

λ: $\quad u_B(x_B, y_B) = 0$ (SINCE $\lambda > 0$) $\quad$ i.e., $2 - \frac{1}{8}(4)^2 = 0$ ok

(b) FROM $\lambda = 4$ WE INFER THAT REDUCING u_B BY ONE UNIT WILL ALLOW u_A TO BE INCREASED BY 4 UNITS.

(c) Z: $(4,1)$ AND $(0,7)$, $MRS_A = 1/4$, $MRS_B = 1$.

Z': $(3, 1\frac{1}{2})$ AND $(1, 6\frac{1}{2})$, $MRS_A = 1/2$, $MRS_B = 3/4$.

FROM Z TO Z' WE HAVE $\Delta x_A = -1$ AND $\Delta y_A = 1/2$;

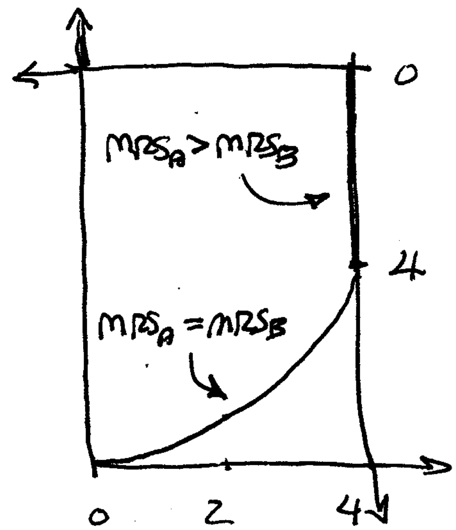
SINCE $\Delta y_A > MRS_A$ AT Z AND $\Delta y_A \geqq MRS_A$ AT Z', A PREFERS Z' TO Z.

SIMILARLY, $\Delta x_B = 1$ AND $\Delta y_B = -1/2$; SINCE $\Delta y_B < 0$ AND $|\Delta y_B| < MRS_B$ AT BOTH Z AND Z', B PREFERS Z' TO Z.

(d) INTERIOR:

$$MRS_A = MRS_B \; ; \quad \text{i.e.,} \quad \frac{y_A}{x_A} = 1 - \frac{1}{4} x_B = 1 - \frac{1}{4}(4 - x_A) = \frac{1}{4} x_A$$

$$\text{i.e.,} \quad y_A = \frac{1}{4} x_A^2$$



MORE GENERALLY:

IF $x_1^1 > 0$ AND $x_2^2 > 0$:

$$MRS_A \geqq MRS_B : \quad \frac{y_A}{x_A} \geqq \frac{1}{4} x_A$$

$$\text{i.e.,} \quad y_A \geqq \frac{1}{4} x_A^2$$

$\therefore \; x_B = 0, \; x_A = 4 \Rightarrow y_A \geqq 4.$

IF $x_2^1 > 0$ AND $x_1^2 > 0$:

$$MRS_A \leqq MRS_B : \quad \frac{y_A}{x_A} \leqq \frac{1}{4} x_A \; ; \quad \text{i.e.,} \quad y_A \leqq \frac{1}{4} x_A^2 .$$

BUT THIS IS INCONSISTENT WITH $y_B = 0, \; y_A = 8, \; x_A \leqq 4$.

$\therefore$ NO PARETO ALLOCATIONS WITH $y_B = 0$ OR $x_A = 0$.

(e) SINCE THE INITIAL ALLOCATION IS PARETO EFFICIENT, WE KNOW (BY THE SECOND WELFARE THEOREM) THAT IT CAN BE SUPPORTED AS A WALRASIAN EQUILIBRIUM. THE EFFICIENCY PRICES (LAGRANGE VALUES) σ_x AND σ_y IN (a) WILL SERVE AS EQUILIBRIUM PRICES: $p_x = 6, \; p_y = 4$. SINCE $(\hat{x}_i, \hat{y}_i) = (\mathring{x}_i, \mathring{y}_i)$ FOR EACH i, WE OBVIOUSLY HAVE $p_x \hat{x}_i + p_y \hat{y}_i = p_x \mathring{x}_i + p_y \mathring{y}_i$ AS WELL. AND EACH i IS MAXIMIZING UTILITY: $MRS_A = \frac{p_x}{p_y}$ AND $MRS_B < \frac{p_x}{p_y}$ WITH $x_B = 0$.

OF COURSE, ANY OTHER PRICE-LIST THAT SATISFIES $\frac{p_x}{p_y} = \frac{3}{2}$ WILL ALSO BE AN EQUILIBRIUM PRICE-LIST. IF $\frac{p_x}{p_y} \neq \frac{3}{2}$, A WILL NOT CHOOSE (4,6) AND MARKETS WILL NOT CLEAR.

(f) $(\mathring{x}_A, \mathring{y}_A) = (4, 0)$, $(\mathring{x}_B, \mathring{y}_B) = (0, 8)$:

$$x_A = \frac{1}{2p_x} w_A = \frac{1}{2p_x}(4p_x) = 2$$

x_B SATISFIES $MRS_B = p_x$ (LET $p_y \equiv 1$);

i.e., $1 - \frac{1}{4} x_B = p_x$

i.e., $\frac{1}{4} x_B = 1 - p_x$; $x_B = 4 - 4p_x$

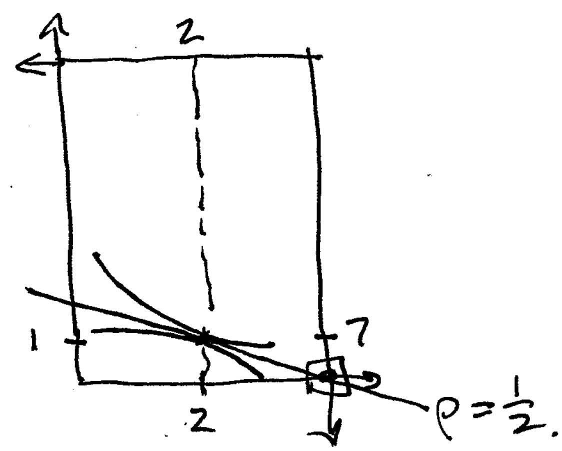
$\therefore x_A + x_B = 2 + 4 - 4p_x = 6 - 4p_x$

MARKET-CLEARING: $6 - 4p_x = \mathring{x} = 4$

i.e., $4p_x = 2$; $\boxed{p_x = \frac{1}{2},\ p_y = 1}$

$$\left.\begin{aligned} x_A &= 2, \\ y_A &= p_x(\mathring{x}_A - x_A) = (\tfrac{1}{2})(2) = 1 \end{aligned}\right\} \quad (x_A, y_A) = (2, 1)$$

$$\left.\begin{aligned} x_B &= 4 - 4p_x = 4 - 2 = 2 \\ y_B &= 8 - p_x x_B = 8 - (\tfrac{1}{2})(2) = 7 \end{aligned}\right\} \quad (x_B, y_B) = (2, 7).$$



(2) (a) SUPPOSE $x \in A \times B$ AND $y \in A \times B$ AND $\lambda \in [0,1]$. LET $z = \lambda x + (1-\lambda) y$; WE MUST SHOW THAT $z \in A \times B$.

BY DEFINITION, $x \in A \times B$ MEANS THERE EXIST $x_a \in A$ AND $x_b \in B$ SUCH THAT $x = (x_a, x_b)$. SIMILARLY, THERE ARE $y_a \in A$ AND $y_b \in B$ SUCH THAT $y = (y_a, y_b)$.

WE HAVE

$$\begin{aligned} z = \lambda x + (1-\lambda) y &= \lambda (x_a, x_b) + (1-\lambda)(y_a, y_b) \\ &= (\lambda x_a + (1-\lambda) y_a,\ \lambda x_b + (1-\lambda) y_b) \\ &= (z_a, z_b), \end{aligned}$$

WHERE $z_a := \lambda x_a + (1-\lambda) y_a$ AND $z_b := \lambda x_b + (1-\lambda) y_b$.

SINCE $x_a, y_a \in A$ AND A IS CONVEX, WE HAVE $z_a \in A$; SIMILARLY, WE HAVE $z_b \in B$; i.e., $z = (z_a, z_b) \in A \times B$. ||

(b) SUPPOSE $\{x(n)\}$ IS A SEQUENCE IN $A \times B$ THAT CONVERGES TO $\bar{x}$. WE MUST SHOW THAT $\bar{x} \in A \times B$.

FOR EACH n, WE HAVE $x(n) = (x_a(n), x_b(n))$, WHERE $x_a(n) \in A$ AND $x_b(n) \in B$. MOREOVER, $x_a(n) \to \bar{x}_a$ AND $x_b(n) \to \bar{x}_b$, WHERE $\bar{x} = (\bar{x}_a, \bar{x}_b)$. SINCE A AND B ARE CLOSED, WE HAVE $\bar{x}_a \in A$ AND $\bar{x}_b \in B$ — i.e., $\bar{x} = (\bar{x}_a, \bar{x}_b) \in A \times B$. ||

3 (a) DEFINE $f: [0,M]^3 \to [0,M]^3$ AS FOLLOWS:

$$p_1' = f_1(p_1, p_2, p_3) = \lambda_{12} p_2 + \lambda_{13} p_3$$

$$p_2' = f_2(p_1, p_2, p_3) = \lambda_{21} p_1 + \lambda_{23} p_3 + \lambda_{20} M$$

$$p_3' = f_3(p_1, p_2, p_3) = \lambda_{31} p_1 + \lambda_{32} p_2 .$$

WITH THE GIVEN RESTRICTIONS ON THE λ_{ij}'s, EACH p_i' IS A CONVEX COMBINATION OF p_1, p_2, p_3, AND M; THEREFORE (SINCE $[0,M]$ IS A CONVEX SET) $p_i' \in [0,M]$ IF EACH $p_1, p_2, p_3 \in [0,M]$. IN OTHER WORDS,

$$p \in [0,M]^3 \Rightarrow f(p) \in [0,M]^3 .$$

AS IN 2

SINCE $[0,M]^3$ IS NONEMPTY, CLOSED, AND CONVEX, AND EACH f_i IS CLEARLY CONTINOUS (IN FACT, LINEAR), BROUWER'S THEOREM GUARANTEES A $\hat{p} \in [0,M]^3$ FOR WHICH $f(\hat{p}) = \hat{p}$ — i.e., $\hat{p}' = \hat{p}$.

(b) IF $p_1 = p_2 \neq 0$, THEN $p_3' = (\lambda_{31} + \lambda_{32}) p_1 < \frac{1}{2} p_1 \neq p_1$. THEREFORE IF $p_1' = p_1$ AND $p_2' = p_2$, WE HAVE $p_3' < p_1', p_2'$. MOREOVER, $p_2' > 0$ (BECAUSE $\lambda_{20} > 0$), SO WE CANNOT HAVE $p_1 = p_2 = p_3 = 0$.

(4) (a) SUPPOSE $\hat{x}$ IS NOT A SOLUTION OF (*) — i.e., THERE IS AN $\tilde{x} \in X$ FOR WHICH $u_i(\tilde{x}) \geqq u_i(\hat{x})$, $i = 2, \ldots, n$, AND FOR WHICH $u_1(\tilde{x}) > u_1(\hat{x})$. THUS, $\tilde{x}$ IS A PARETO IMPROVEMENT ON $\hat{x}$ — i.e., $\hat{x}$ IS NOT PARETO EFFICIENT.

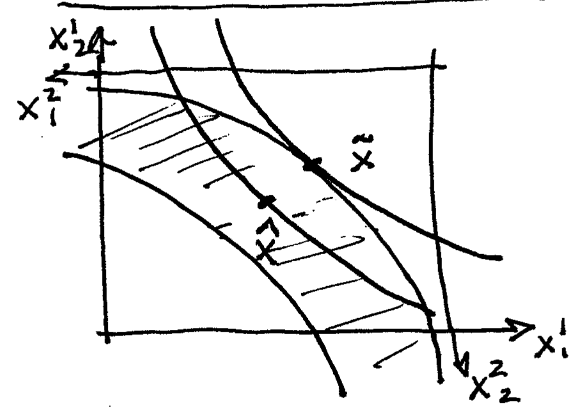
(b) LET $n = 2$, AND LET X BE THE SET OF FEASIBLE ALLOCATIONS OF TWO GOODS TO THE TWO PEOPLE, SAY $(\mathring{x}_1, \mathring{x}_2) = (6,6)$. LET $\hat{x}$ BE THE ALLOCATION $((3,3),(3,3))$, AND LET

$$u_1(x) = x_1 x_2 \quad \text{AND} \quad u_2(x) = \begin{cases} 9 & \text{IF } 4 \leqq x_1 x_2 \leqq 16 \\ x_1 x_2 & \text{OTHERWISE.} \end{cases}$$

IT'S CLEAR THAT $\hat{x}$ IS A SOLUTION OF (*), BUT $\tilde{x}$ IS A PARETO IMPROVEMENT ON $\hat{x}$, WHERE $\tilde{x}^1 = (4,4)$ AND $\tilde{x}^2 = (2,2)$:

$$u_1(\tilde{x}^1) = 16 > u_1(\hat{x}^1)$$
$$u_2(\tilde{x}^2) = 4 = u_2(\hat{x}^2).$$

DIAGRAMMATICALLY:



NOTE: u^2 HERE IS NOT CONTINUOUS. WHILE CONTINUITY IS IRRELEVANT FOR THIS QUESTION, IT IS POSSIBLE TO CONVERT THIS u_2 TO A $\tilde{u}_2$, VIA A MONOTONE TRANSFORM, WHERE $\tilde{u}_2$ IS CONTINUOUS:

$$\tilde{u}_2(x) = \begin{cases} u_2(x) + 5 = x_1 x_2 + 5, & \text{IF } x_1 x_2 < 4 \\ u_2(x) = 9, & \text{IF } 4 \leqq x_1 x_2 \leqq 16 \\ u_2(x) - 7 = x_1 x_2 - 7, & \text{IF } x_1 x_2 > 16 \end{cases}$$