Economics 501B Midterm Exam
University of Arizona
Fall 2009

1. Al and Bill have identical preferences for the two goods X and Y, represented by the utility function $u(x,y) = 2\alpha\sqrt{x} + 2\sqrt{y}$. Note that

$$u_x = \alpha/\sqrt{x}, \qquad u_y = 1/\sqrt{y}, \qquad \text{and } MRS = \alpha\sqrt{y/x}.$$

There are $\mathring{x}$ units of X and $\mathring{y}$ units of Y to be allocated to Al and Bill.

(a) Write down the maximization problem that characterizes the Pareto allocations and derive the first-order conditions that characterize the interior solutions of the maximization problem. Determine the efficiency (decentralizing) prices as functions of the parameters $\mathring{x}$, $\mathring{y}$, and α.

(b) Assume that $\alpha = 1$, $\mathring{x} = 36$, and $\mathring{y} = 64$. Determine the set of all Pareto allocations and draw the set in an Edgeworth box diagram.

(c) Assume that $\alpha = 1$, $(\mathring{x}_A, \mathring{y}_A) = (21, 0)$, and $(\mathring{x}_B, \mathring{y}_B) = (15, 64)$. Use the First Welfare Theorem and the efficiency prices you found in part (b) to obtain the Walrasian equilibrium prices and allocation. Be sure to make clear what the role of the First Welfare Theorem is in your determination of the equilibrium.

2. Let S denote the unit simplex in $\mathbb{R}^l$ and let $\zeta : S \to \mathbb{R}^l$ be a market demand function — *i.e.*, ζ is a continuous function that satisfies Walras' Law, $\mathbf{p} \cdot \zeta(\mathbf{p}) = 0$ for all $\mathbf{p} \in S$. A price-list $\mathbf{p} \in S$ is an equilibrium of ζ if for every k, $\zeta_k(\mathbf{p}) \leqq 0$ and $p_k\zeta_k(\mathbf{p}) = 0$.

(a) Prove that if $\zeta_k(\mathbf{p}) \leqq 0$ for each k, then $\mathbf{p}$ is an equilibrium of ζ.

(b) For each $k = 1, \ldots, l$, define a real-valued function $M_k : S \to \mathbb{R}_+$ as $M_k(\mathbf{p}) = \max\{0, \zeta_k(\mathbf{p})\}$, and define the function $f : S \to S$ as follows:

$$f(\mathbf{p}) = \frac{1}{\sum_1^l [p_k + M_k(\mathbf{p})]}[\mathbf{p} + M(\mathbf{p})].$$

Prove that if $\mathbf{p}$ is a fixed point of f, then $M(\mathbf{p}) \cdot \zeta(\mathbf{p}) = 0$. Then use the result in (a) to prove that if $M(\mathbf{p}) \cdot \zeta(\mathbf{p}) = 0$, then $\mathbf{p}$ is an equilibrium of ζ.

(c) Use Brouwer's Fixed Point Theorem to prove that f has a fixed point.

3. Ann and Bev have identical preferences for oranges and orange juice (OJ), represented by the utility function $u(x, y) = x^3y$, where x denotes bushels of oranges and y denotes gallons of OJ. Between the two of them, they own 75 bushels of oranges and 5 gallons of OJ. There is also a machine that can turn oranges into OJ according to the following production function: the first five bushels used to produce OJ each yield 3 gallons of juice; the next 20 bushels each yield 2 gallons of juice; and any additional bushels each yield only one gallon of juice. Formally,

$$q = f(z) = \begin{cases} 3z, & z \leqq 5 \\ 5 + 2z, & 5 \leqq z \leqq 25 \\ 30 + z, & z \geqq 25, \end{cases}$$

where z denotes the number of bushels of oranges used as input in the production process and q denotes the number of gallons of OJ obtained as output.

(a) Draw a diagram depicting carefully the consumption possibilities set for this economy – *i.e.*, the set of all (x, y) combinations of oranges and OJ that can be achieved.

(b) State the marginal condition(s) on production and consumption that characterize interior Pareto allocations.

(c) Use the conditions in (b) to determine whether any of the following consumption allocations are consistent with Pareto efficiency:

(c1) $(x_A, y_A) = (35, 10)$ and $(x_B, y_B) = (35, 10)$
(c2) $(x_A, y_A) = (48, 21)$ and $(x_B, y_B) = (16, 7)$
(c3) $(x_A, y_A) = (45, 30)$ and $(x_B, y_B) = (15, 10)$

For any of the above consumption allocations that are Pareto optimal, determine the associated decentralizing prices. For any that aren't Pareto optimal, find a Pareto improvement. (The Pareto improvement need not be Pareto optimal, just a Pareto improvement.)

(d) Now assume that Ann owns 55 of the bushels of oranges and all 5 gallons of OJ; Bev owns the remaining 20 bushels of oranges and no OJ; and Ann owns the OJ-producing machine and any profit that might accrue from using the machine in the production process. Verify that there is a Walrasian equilibrium in which the prices are $p_x = \$2$ and $p_y = \$1$ and 35 gallons of OJ are produced. Determine the equilibrium level of input usage, Ann's profit, and both Ann's and Bev's consumption bundles. Verify that both markets clear.