1. \[ P = 100 - q, \quad q = q_1 + q_2, \quad C_1(q_i) = 10q_i, \quad C_2(q_2) = 40q_2. \]

(a) \[ MR_1 = MC_1: \quad 100 - q_2 - 2q_1 = 10 \quad 2q_1 + q_2 = 90 \quad q_1 = 45 - \frac{1}{2}q_2 \]
\[ MR_2 = MC_2: \quad 100 - q_1 - 2q_2 = 40 \quad q_1 + 2q_2 = 60 \quad q_2 = 30 - \frac{1}{2}q_1 \]

NE: \[ q_1 = 45 - \frac{1}{2}(30 - \frac{1}{2}q_1) = 30 + \frac{1}{4}q_1; \quad \frac{3}{4}q_1 = 30 \]
\[ q_1 = 40, \quad q_2 = 10; \quad q = 50, \quad P = 50 \]
\[ \pi_1 = 40(50 - 10) = 1600, \quad \pi_2 = 10(50 - 40) = 100. \]

(b) If they collude:

All production by Firm 1 (lower unit cost throughout).

\[ MR_1 = MC_1: \quad 100 - 2q_1 = 10; \]
\[ \therefore q_1 = 45, \quad q_2 = 0; \quad P = 55; \quad \pi_1 = 45(55 - 10) = 2025, \quad \pi_2 = 0. \]

(c) \[ P = MC_1 = 10 (\text{lowest } MC \text{ firm}); \quad q_2 = 0 (MC_2 > P); \quad q_1 = 90. \]
\[ \pi_1 = \pi_2 = 0. \]

(d) \[ U(x,y) = y + 100x - 5x^2; \quad MRS = 100 - 10x. \]

At equilibrium: \[ MRS = P; \quad i.e., \quad 100 - 10x = 50; \quad x = 10 - \frac{1}{10}P. \]

Total demand by the 10 consumers \[ x = 10x = 100 - P, \] as above.

Consumer surplus in (a):

\[ C_1: \quad \frac{1}{2} \times 125 \times 50 = 3125, \quad C_2: \quad \frac{1}{2} \times 125 \times 50 = 3125. \]

Consumer surplus in (c):

\[ C_1: \quad \frac{1}{2} \times 40 \times 10 = 200, \quad C_2: \quad \frac{1}{2} \times 40 \times 10 = 200. \]

Additionally,
\[ U(y - 250, 5) = 9 + 250 + 500 - 125 = 9 + 125 = 134, \quad \frac{1}{3} \]
\[ U(y - 10, 9) = 9 + 90 + 900 - 405 = 450. \]
(a) A move from (a) to (c) will increase each consumer's CS (and utility) by $280 = 405 - 125$. But it will also decrease the firms' profits by 1800 (firm 1) and 100 (firm 2). We will therefore have to transfer at least 170 from each consumer, to compensate the firms; but no more than 280 from each consumer.

Transfer $170 \leq t \leq 280$ from each consumer, with firms receiving $s_1 \geq 1600$ and $s_2 \geq 100$.

(b) If $c_2(q_2) = 70q_2$:

$MR_2 = MC_2: 100 - q_1 - 2q_2 = 70, q_1 + 2q_2 = 30, q_2 = 15 - \frac{1}{2}q_1$.

NE: if $q_1 \geq 30$, then $q_2 = 0$; if $q_2 = 0$, then $q_1 = 45 \geq 30$.

So NE is $(q_1, q_2) = (45, 0)$; $P = 55$, as in (b).

[Diagram showing intersect at $(45, 0)$]
2. $u_A(x,y_A)$ and $u_B(x,y_B)$

(a) **Samuelson marginal condition**: $MRS_A + MRS_B = MC.$
Here $MC = 0,$ so $MRS_A + MRS_B = 0$ — i.e., $MRS_A = -MRS_B.$

(b) $\max_{x,y_A,y_B \geq 0} \lambda_A u_A(x,y_A)$ s.t. $y_A + y_B \leq y$:

\[ u_A(x,y_A) \geq u_B(x,y_B) \geq u_B : \lambda_B \]

\textbf{Interior FOC:} \quad \exists \delta, \lambda_B \geq 0 \text{ s.t.}

- $x: \lambda_A u_A = -\lambda_B u_Bx \quad \text{i.e.,} \quad \lambda_A u_A + \lambda_B u_Bx = 0$
- $y_A: \lambda_A u_Ay = \sigma$
- $y_B: \sigma = \sigma - \lambda_B u_By \quad \text{i.e.,} \quad \lambda_B u_By = \sigma$

When $x < 75$ we have $u_A(x,y_A), u_B(x,y_B) > 0,$

so also $\lambda_B \sigma > 0.$

\textbf{Combining the FOC equations:}

\[ \sigma \frac{u_A}{u_B} + \sigma \frac{u_Bx}{u_by} = 0 \quad \text{i.e.,} \quad MRS_A + MRS_B = 0. \]

(c) $u_A(x,y_A) = y_A - (x-75)^2, \quad u_B(x,y_B) = y_B - \frac{1}{2}(x-69)^2$

\[ MRS_A = 150 - 2x, \quad MRS_B = 69 - x \]

$MRS_A + MRS_B = 219 - 3x = 0 \quad \Rightarrow \quad x = 73; \quad y_A + y_B = y.$

\[(x = 73) \quad u_A(73, y_A + t) = y_A + t - 4, \quad u_A(75, y_A) = y_A. \]

\[ u_B(73, y_B - t) = y_B - t - 8, \quad u_B(75, y_B) = y_B - 18. \]

\textbf{Any transfer that satisfies} $4 \leq t \leq 10 \text{ will make both Anna and Bob better off at } x = 73.$

They \textbf{were at} $x = 75 \text{ with no transfer, because } x = 73 \text{ is Pareto efficient, they cannot both do better.}$
(e) \( C(x) = 6(35-x) \) for \( x \geq 35 \). \( MC = -6 \).

\[ \text{MRS}_A + \text{MRS}_B = MC \]

\[ (150 - 2x) + (69 - x) = -6 \]

\[ 219 - 3x = -6 \]

\[ 3x = 225 \]

\[ x = 75 \]
\( u_A(x_0, x_H, x_L) = x_0 + 5x_H - 3x_H^2 + 3x_L - 3x_L^2 \)
\( u_B(x_0, x_H, x_L) = x_0 + 5x_H - 4x_H^2 + 3x_L - 2x_L^2 \)
\[ x_0 = (6, 4, 2), \quad x_B = (6, 10, 8), \quad \therefore x = (12, 14, 10). \]

\[ MRS_{AH} = 5 - 0.6x_AH \quad MRS_{AL} = 3 - 0.6x_AL \]
\[ MRS_{BH} = 5 - 0.8x_BH \quad MRS_{BL} = 3 - 0.4x_BL \]

(4.) \[ MRS_{AH} = MRS_{BH} : \quad 5 - 0.6x_AH = 5 - 0.8x_BH \quad \therefore 6x_AH = 8x_BH \]
\[ \therefore x_AH = \frac{8}{6}, \quad x_BH = \frac{6}{8}. \]

\[ MRS_{AL} = MRS_{BL} : \quad 3 - 0.6x_AL = 3 - 0.4x_BL \quad \therefore 6x_AL = 4x_BL \]
\[ \therefore x_AL = \frac{4}{6}, \quad x_BL = \frac{6}{4}. \]

\( x_{A0}, x_{B0} \) need only satisfy \( x_{A0} + x_{B0} = x_0 = 12 \).

Note that \( MRS_{AH} = MRS_{BH} = \frac{5 - 4.8}{0.8} = 0.2 \) and \( MRS_{AL} = MRS_{BL} = \frac{3 - 2.4}{0.6} = 0.6 \).

(6.) The Arrow-Debreu equilibrium satisfies \( MRS_{iH} = p_i \) and \( MRS_{iL} = p_L \) for \( i=A, B \) (assuming \( p_B = 1 \)), so the results in (2) yield \( p_H = 0.2 \) and \( p_L = 0.6 \).

The results in (2) then also yield the individuals' choices of \( x_{iH} \) and \( x_{iL} \): \( (x_{AH}, x_{AL}) = (8, 4) \) and \( (x_{BH}, x_{BL}) = (6, 6) \).

These choices, along with the prices, yield
\[ x_{A0} = \frac{8}{0.2} - p_H x_{AH} - p_L x_{AL} = 6 \cdot 2(8 - 4) - 0.6(4 - 2) = 6 - 8 - 1.2 = 4 \]
\[ x_{B0} = \frac{6}{0.2} - p_H x_{BH} - p_L x_{BL} = 6 \cdot 2(6 - 10) - 0.6(6 - 8) = 6 + 8 + 1.2 = 8 \]
(c) Let $S_i$ denote $i$'s saving. Equilibrium satisfies $S_A + S_B = 0$ and $MRS_{iH} + MRS_{iL} = \frac{1}{1+\tau}$ for each $i$ (this is derived below, although it was not required).

Note that $x_{Ah} = x_{Ah} + (1+\tau)S_A$ and $x_{Al} = x_{Al} + (1+\tau)S_A$; and similarly for B. Writing $\bar{z}$ for $(1+\tau)S_0$ to simplify notation, we have

$$MRS_{Ah} + MRS_{Al} = 5 - 0.6(4+2\bar{z}) - 0.6(2+2\bar{z}) + 3$$
$$= 8 - 2.4 - 1.2 - 1.2 \bar{z}_A = 4.4 - 1.2 \bar{z}_A$$

$$MRS_{Bh} + MRS_{Bl} = 5 - 0.8(10+2\bar{z}) + 3 - 0.4(8+2\bar{z})$$
$$= 8 - 8 - 3.2 - 1.2 \bar{z}_B = -3.2 - 1.2 \bar{z}_B$$

At equilibrium we have $S_B = -S_A$ (and $\therefore$ $2_B = -2_A$) and

$MRS_{Ah} + MRS_{Al} = \frac{1}{1+\tau} = MRS_{Bh} + MRS_{Bl}$

i.e. $4.4 - 1.2 \bar{z}_A = -3.2 - 1.2 (-2_A)$

i.e. $2.4 \bar{z}_A = 7.6; \quad \therefore \bar{z}_A = \frac{76}{24} = \frac{19}{6} = 3.16$.

$\therefore MRS_{Ah} + MRS_{Al} = 4.4 - (1.2)(\frac{19}{6}) = 4.4 - (1.2)(\frac{14}{6}) = 4.4 - \frac{19}{6} = 0.6$.

$MRS_{Bh} + MRS_{Bl} = -3.2 + 1.2 \bar{z}_A = -3.2 + (\frac{4}{3})\bar{z}_A = -3.2 + \frac{14}{6} = 0.6$.

$\therefore \frac{1}{1+\tau} = 0.6; \quad 1+\tau = \frac{1}{0.6} = \frac{5}{3} = 1.66 \therefore \sqrt{1.66} = 1.3$.

Solving for $S_A$ and $S_B$:

$(1+\tau)S_A = 2_A; \quad i.e. \frac{5}{3}S_A = \frac{19}{6}; \quad \therefore S_A = (\frac{3}{5})(\frac{19}{6}) = (\frac{6}{5})(\frac{19}{6}) = 1.9$.

$S_A = 1.9, \quad S_B = -1.9$

$x_{Ah} = 4 + 2A = 7.6; \quad x_{Al} = 2 + 2A = 5.6; \quad \{\text{Note how these compare with (a)!}\}$

$x_{Bh} = 10 + 2B = 6.6; \quad x_{Bl} = 8 + 2B = 4.6$

$x_{A0} = x_{A} - S_A = 6 - 1.9 = 4.1; \quad x_{B0} = x_{B} - S_B = 6 + 1.9 = 7.9$. 
(d) Two securities:

\[ d_1 = \left[ \frac{1}{1+r} \right], \text{ price is } 1; \quad d_2 = \left[ 0 \right], \text{ price is } p. \]

Note that we could instead have defined \( d_1 \) by \( d_1 = \left[ \frac{1}{1} \right] \), with price \( q \). Then the interest rate \( r \) would be defined by \( q = \frac{1}{1+r} \).

Since the securities span the space of state-contingent returns, \( \mathbb{R}^2 \), the equilibrium allocation here will be the Arrow-Debreu allocation, and the equilibrium \( r \) and \( p \) will be related to the Arrow-Debreu prices \( P_U \) and \( P_L \) as follows:

- Price of \( d_1 \): \( 1 = (1+r)P_U + (1+r)P_L = (1+r)(P_U+P_L) = (1+r)(.8) \)
- Price of \( d_2 \): \( p = (1)P_U + (0)P_L = P_U = .2. \)

From the equation for \( d_1 \)'s price we have \( \frac{1}{1+r} = .8 \) \( \Rightarrow r = \frac{1}{8} = 25\% \).

So we have: \[ \text{Interest rate is } r = \frac{1}{8} = 25\%, \]
\[ \text{Price of insurance is } p = .2. \]

If we had instead defined \( d_1 \) as \( d_1 = \left[ 1 \right] \), with price \( q \), we would have \( q = (1)P_U + (1)P_L = .2 + .6 = .8 \), so we again have \( \frac{1}{1+r} = .8 \) and therefore \( r = \frac{1}{8} \).

The easiest way to determine the saving and insurance choices is to use the Arrow-Debreu quantities, as follows:
**Saving:** The only way to augment \( x_L \) is via saving; e.g., \( x_L = \bar{x}_L + (1+r)\bar{s} \) or \( \bar{s} = \frac{1}{1+r} (x_L - \bar{x}_L) \).

**Saving by A:** \( S_A = \frac{1}{1+r} (x_{AL} - \bar{x}_{AL}) = 0.8(4-2) = 1.6 \).

**Saving by B:** \( S_B = \frac{1}{1+r} (x_{BL} - \bar{x}_{BL}) = 0.8(6-8) = -1.6 \).

Thus, A saves 1.6, which yields 2 tomorrow in each state; B borrows 1.6, repaying 2 tomorrow in each state.

**Insurance:** With no insurance, consumption in state H would be \( x_H = \bar{x}_H + (1+r)\bar{s} \), the same as in state L.

But insurance augments this: buying \( \bar{s} \) units of insurance provides \( x_H = \bar{x}_H + (1+r)\bar{s} + \bar{s} \), i.e., \( \bar{s} = x_H - (\bar{x}_H + (1+r)\bar{s}) \).

**Insurance by A:** \[ S_A = x_{AH} - (\bar{x}_{AH} + (1+r)S_A) \]
\[ = 8 - (4+2) = 8 - 6 = 2 \]

**Insurance by B:** \[ S_B = x_{BH} - (\bar{x}_{BH} + (1+r)S_B) \]
\[ = 6 - (10-2) = 6 - 8 = -2 \]

Thus, A buys 2 units of insurance, which are sold by B, at price \( p = 0.2 \).

**Check \( x_{AO} \) and \( x_{BO} \) against Arrow-Debreu values:**

\( x_{AO} = x_{AO} - S_A - pS_A = 6 - 1.6 - (0.2)(2) = 6 - 2 = 4 \)

\( x_{BO} = x_{BO} - S_B - pS_B = 6 - (-1.6) - (0.2)(-2) = 6 + 2 = 8 \)

The same as the Arrow-Debreu values.
(8). The allocation in (d) is Pareto efficient, the one in (c) is not: the one in (d) satisfies the equations in (c), the one in (c) does not. More generally, the explanation is that there are complete securities markets in (d) — the securities span the space of state-contingent returns, so that individuals can achieve independent choices of \( x_H \) and \( x_L \). In (c) that's not possible: \( x_H - \hat{x}_H \) and \( x_L - \hat{x}_L \) must be the same, namely \( (1+r)s \).

Derivation of the marginal condition for utility maximization with only a credit market, as in (c):

[This was not required]

Define \( \bar{u}(s) := u(0, x_H(s), x_L(s)) \)

\[ = u\left(s - \hat{x}_H, (1+r)s, (1+r)s\right). \]

Interior FOC: \( \frac{\partial \bar{u}}{\partial s} = 0 \)

i.e., \( (1)u_0 + (1+r)u_H + (1+r)u_L = 0 \)

i.e., \( (1+r)(u_H + u_L) = u_0 \)

i.e., \( \frac{u_H}{u_0} + \frac{u_L}{u_0} = \frac{1}{1+r} \)

i.e., \( MRS_H + MRS_L = \frac{1}{1+r}. \)