BUG-SPRAY EXAMPLE

(A PUBLIC GOOD)

A HOMEOWNERS' ASSOCIATION WITH n MEMBERS (ALL THE AFFECTED HOMEOWNERS).

x: GALLONS OF MOSQUITO SPRAY PER WEEK.

y_i: CONSUMPTION OF "OTHER GOODS" (e.g., \$) BY i.

HOMEOWNER i'S CHARACTERISTICS:

UTILITY FUNCTION $\quad u^i(x, y_i)$

ENDOWMENT $\quad \mathring{y}_i \qquad [\mathring{x}_i = 0, \forall i]$.

PRODUCTION TECHNOLOGY:

$$x = f(z) \quad \text{OR} \quad z = C(x).$$

FEASIBILITY:

$$\sum_1^n y_i + z \leqq \sum_1^n \mathring{y}_i$$

$$x \leqq f(z)$$

MARGINAL CONDITIONS FOR PARETO EFFICIENCY

(w/ ONE PUBLIC GOOD, ONE PRIVATE GOOD)

$$\max \ \lambda_1 u^1(x, y_1)$$

s.t. $x, z, y_1, \dots, y_n \geqq 0$

AND TO

$$\sum y_i + z \leqq \overset{o}{y} \qquad : \sigma$$

$$x \leqq f(z) \qquad : \mu$$

$$\{\text{OR } C(x) \leqq z, \text{ WHERE } C(x) := f^{-1}(z)\}$$

$$u^i(x, y_i) \geqq e_i, \ i = 2, \dots, n \qquad : \lambda_i$$

FIRST-ORDER CONDITIONS:

x: $\lambda_1 u^1_x \leqq \mu - \sum_2^n \lambda_i u^i_x$ "=" IF $x > 0$

i.e., $\sum \lambda_i u^i_x \leqq \mu$

z: $0 \leqq \sigma - \mu f'(z)$ "=" IF $z > 0$

i.e., $\dfrac{\mu}{\sigma} \leqq \dfrac{1}{f'(z)} = MC$

y_1: $\lambda_1 u^1_y \leqq \sigma$ "=" IF $y_1 > 0$

y_i: $0 \leqq \sigma - \lambda_i u^i_y$ "=" IF $y_i > 0$

i.e., $\lambda_i u^i_y \leqq \sigma$

ASSUMING AN INTERIOR SOLUTION
($\therefore$ ALL ARE EQUATIONS) AND COMBINING:

$$\sum \lambda_i u_x^i = \mu \quad \text{AND} \quad \lambda_i = \frac{\sigma}{u_y^i}, \ \forall i$$

$$\therefore \quad \sum \sigma \frac{u_x^i}{u_y^i} = \mu$$

$$\text{i.e.,} \quad \sigma \sum \frac{u_x^i}{u_y^i} = \mu$$

$$\text{i.e.,} \quad \sum MRS^i = \frac{\mu}{\sigma} = \frac{1}{f'(z)} = MC$$

$$\text{i.e.,} \quad \boxed{\sum_{i=1}^{n} MRS^i = MC},$$

THE SAMUELSON MARGINAL CONDITION
FOR EFFICIENCY WITH A PUBLIC GOOD.

NOTE THAT THE FIRST TWO CONSTRAINTS CAN BE COMBINED INTO A SINGLE CONSTRAINT,

$$\sum_{i=1}^{n} y_i + C(x) \leq \mathring{y}, \quad \text{WHERE} \quad C(x) = f^{-1}(z).$$

IF THERE ARE CONSTANT RETURNS TO SCALE (SAY, $C(x) = \beta x$), THEN THE CONSTRAINT IS

$$\sum_{i=1}^{n} y_i + \beta x \leq \mathring{y}.$$

IF THE PUBLIC GOOD IS COSTLESS [i.e., $C(x) \equiv 0$], AS IN THE "WATER LEVEL" EXAMPLE:

THE ONLY STRUCTURAL CONSTRAINT IS

$$\sum_{i=1}^{n} y_i \leqq \mathring{y}.$$

MARGINAL CONDITIONS:

x: $\sum_1^n \lambda_i u_x^i = 0$ IF $x > 0$.

y_i: $\lambda_i u_y^i \leqq \sigma$, W/ EQUALITY IF $y_i > 0$ $(i = 1, \ldots, n)$.

IF $y_i > 0$ $(\forall i)$, THESE YIELD

$$\sigma \sum_1^n \frac{u_x^i}{u_y^i} = 0$$

i.e., $\boxed{\sum_1^n MRS^i = 0.}$

ALSO:

IF $\sum MRS^i > 0$, THEN PO REQUIRES THAT

$$\exists i : (MRS^i > 0 \text{ AND } y_i = 0).$$

IF $\sum MRS^i < 0$, THEN PO REQUIRES THAT

$$\exists i : (MRS^i < 0 \text{ AND } y_i = 0).$$

(PROVE THIS AS AN EXERCISE.)

The Individual's Choice Problem in the Market, with a Public Good

$$\max_{\xi_i} \; u^i(x, y_i) = u^i(X_{-i} + \xi_i, \, y_i)$$

He chooses ξ_i taking X_{-i} as given.

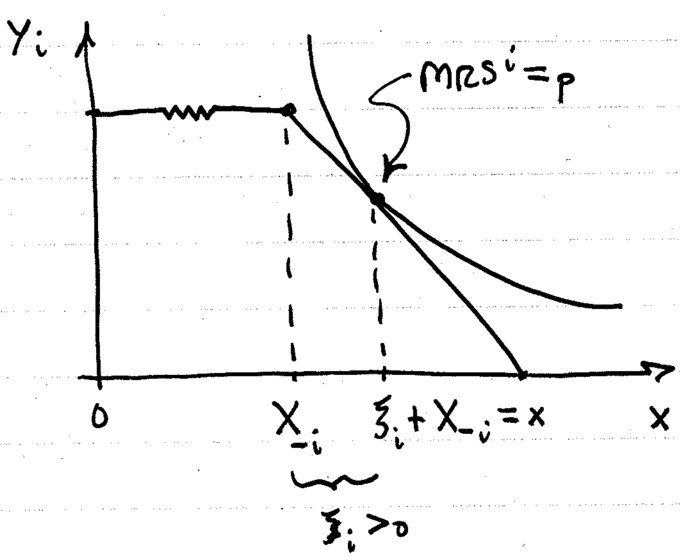
$$\text{s.t. } \xi_i, y_i \geqq 0$$

$$\text{AND TO } \quad p\xi_i + y_i \leqq \mathring{y}_i .$$

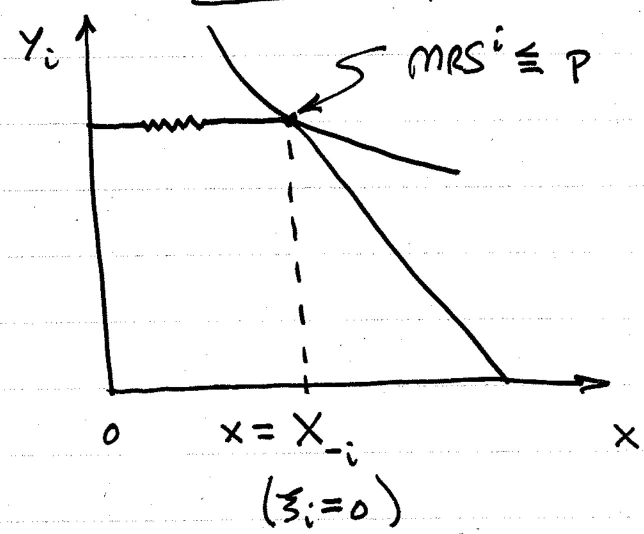
FOC:

$MRS^i \leqq p$, AND $MRS^i = p$ IF $\xi_i > 0$.

IF $\xi_i > 0$:



IF $\xi_i = 0$:



Example:

$c(x) = \beta x$; $p = \beta$ (competitive supply, say)

$$u^i(x, y_i) = y_i + \alpha_i \log x, \quad i = 1, \dots, n$$

$$MRS^i = \frac{\alpha_i}{x}. \qquad \text{Let } A := \sum_{i=1}^{n} \alpha_i$$

and $\alpha^* := \max\{\alpha_i \mid i \in N\}$.

Pareto Efficiency:

$$\sum MRS^i = MC; \quad \text{i.e.,} \quad \sum \frac{\alpha_i}{x} = \beta;$$

$$\text{i.e.,} \quad \frac{1}{x} A = \beta;$$

$$\text{i.e.,} \quad \hat{x} = \frac{A}{\beta}.$$

Market Outcome:

$$MRS^i \leqq p = \beta, \quad \forall i.$$

$$\text{i.e.,} \quad \frac{\alpha_i}{x} \leqq p, \quad \forall i$$

$$\text{i.e.,} \quad x \geqq \frac{\alpha_i}{p}, \quad \forall i.$$

$$MRS^i = p \text{ if } z_i > 0; \quad \text{i.e.,} \quad x = \frac{\alpha_i}{p}.$$

$$\therefore z_i = 0 \text{ if } \alpha_i < \alpha^* = \max\{\alpha_i \mid i \in N\},$$

$$\text{and } x^m = \frac{\alpha^*}{p}.$$

But $\alpha^* \ll A$; $\therefore x^m \ll \hat{x}$.

EXERCISE:

SUPPOSE $\beta = 5$ IS THE (CONSTANT) MC OF THE PUBLIC GOOD, X.

SUPPOSE $n = 11$, i.e., $N = \{1, 2, \ldots, 11\}$, WITH UTILITY FUNCTIONS

$$u^i(x, y_i) = y_i + \alpha_i \log x,$$

WHERE $\alpha_i = i + 9$;

i.e., $\alpha_1 = 10$, $\alpha_2 = 11$, $\alpha_3 = 12$, $\ldots$, $\alpha_{11} = 20$.

ASSUME $\overset{\circ}{y}_i \geqq 100$, $\forall i$.

DETERMINE

(a) THE PARETO EFFICIENT OUTCOME(S) AND EVERYONE'S MRS IN AN EFFICIENT OUTCOME;

(b) THE MARKET OUTCOME, AND EVERYONE'S PURCHASE z_i AND MRS AT THE MARKET OUTCOME.

Exercise Solution:

$\alpha^* = 20, \qquad A = 165.$

Efficiency:

$$\hat{x} = \frac{A}{p} = \frac{165}{5} = 33.$$

$$MRS^i = \frac{\alpha_i}{33}: \quad MRS^1 = \frac{10}{33} \approx .3, \quad MRS^2 = \frac{11}{33} = \frac{1}{3},$$

$$\ldots, MRS^{11} = \frac{20}{33} \approx .60.$$

Market:

$$x^* = \frac{\alpha^*}{p} = \frac{20}{5} = 4.$$

$$MRS^i = \frac{\alpha_i}{4}: \quad MRS^1 = \frac{10}{4} = 2.5, \quad MRS^2 = \frac{11}{4} = 2.75, \ldots$$

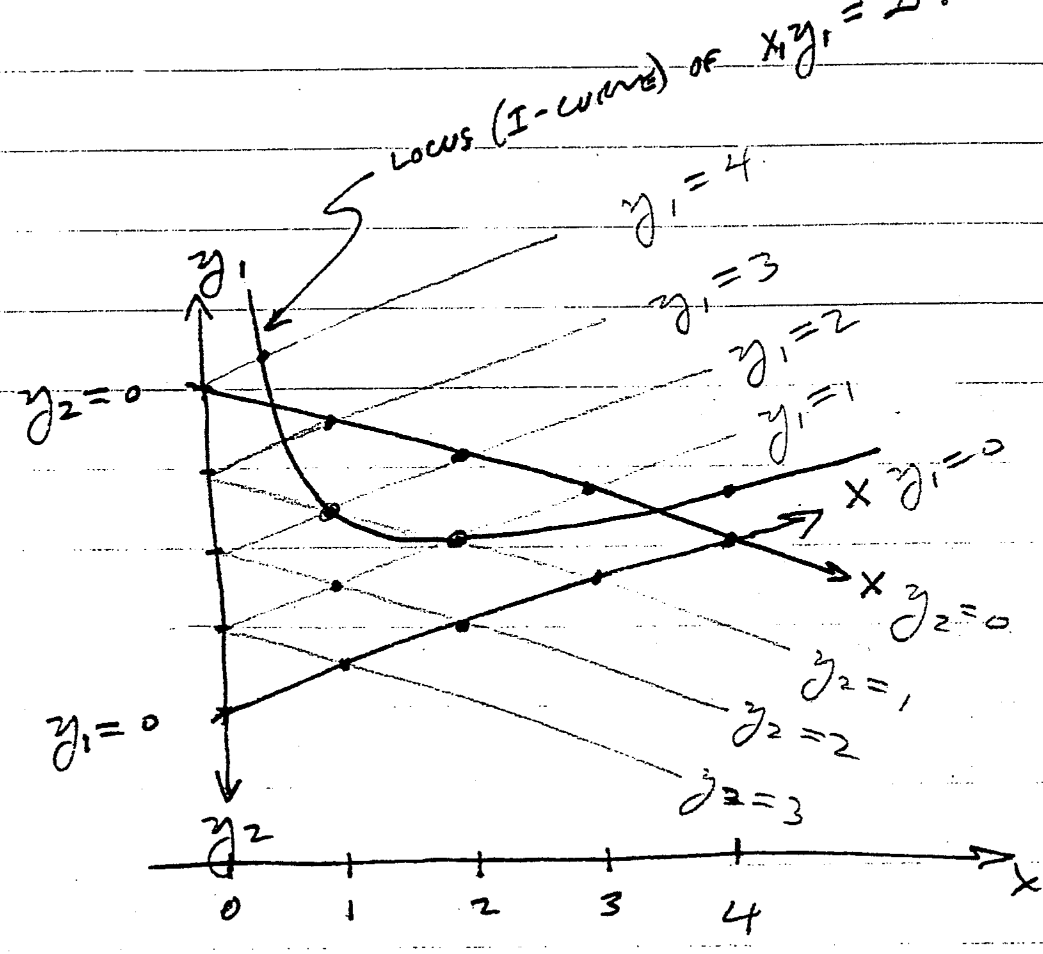
$$\ldots, MRS^{11} = \frac{20}{4} = 5.$$

$$z_1 = z_2 = \cdots = z_{10} = 0; \quad z_{11} = 4.$$

THE KOLM TRIANGLE

(AFTER SERGE-CHRISTOPHE KOLM)

THE COSTLY-PUBLIC-GOOD ANALOGUE OF THE "EDGEWORTH STRIP" IN THE COSTLESS CASE AND THE EDGEWORTH BOX IN THE NO-EXTERNALITIES CASE.



$$\mathring{y}_1 + \mathring{y}_2 = 4$$

$$C(x) = x \quad (MC = 1).$$

$$\therefore \; y_1 + y_2 = 4 - x.$$