Designing a Mechanism with Pareto Efficient Nash Equilibria

(A mechanism to choose the provision level and financing of a public good)

The mechanism has to choose $(x; t_1, \dots, t_n) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Let's allow participants to choose $m_i \in \mathbb{R}$ $(i = 1, \dots, n)$.

A configuration of all n choices is an $m \in \mathbb{R}^n$.

Let's denote the outcome functions, or "rules," as:

$x = \pi(m)$, the "provision function"

$y_i = \mathring{y}_i - t_i$, where $t_i = \tau^i(m)$, i's "tax rule."

If we want the NE to be Pareto efficient, what conditions will the functions $\pi, \tau^1, \dots, \tau^n$ have to satisfy?

Write i's payoff function as $\tilde{u}^i(m) := u^i(\pi(m), \mathring{y}_i - \tau^i(m))$.

At an interior NE we must have $\frac{\partial \tilde{u}^i}{\partial m_i} = 0$. But

$$\frac{\partial \tilde{u}^i}{\partial m_i} = u^i_x \pi_i - u^i_y \tau^i_i, \text{ where } \underbrace{u^i_x = \partial u^i / \partial x,\ u^i_y = \partial u^i / \partial y,\ \pi_i = \partial \pi / \partial m_i,\ \tau^i_i = \partial \tau^i / \partial m_i}.$$

$$\therefore \frac{\partial \tilde{u}^i}{\partial m_i} = 0 \iff \frac{u^i_x}{u^i_y} = \frac{\tau^i_i}{\pi_i}, \text{ i.e., } \boxed{MRS^i = \frac{\tau^i_i}{\pi_i}}$$

Since efficiency requires $\sum MRS^i = MC$ (in the interior), we will have to have

$$\boxed{\sum_{i=1}^{n} \frac{\tau^i_i}{\pi_i} = MC.}$$

IF THE OTHERS' ACTIONS (i.e., THE $(n-1)$-TUPLE $m_{\sim i}$) ARE TAKEN AS GIVEN, THEN THE CHOICE OF m_i BY INDIVIDUAL i IMPLIES A CHOICE OF BOTH x AND y_i. OF COURSE, HIS CHOICE IS CONSTRAINED TO SATISFY SOME KIND OF <u>TRADE-OFF</u> BETWEEN x AND y_i — i.e., THE SET OF (x, y_i) PAIRS AVAILABLE BY VARYING THE CHOICE m_i IS <u>CONSTRAINED</u>, AND THE TRADE-OFF IS THE SLOPE OF THAT CONSTRAINT.

[THE PROTOTYPE FOR THIS IDEA IS THE FAMILIAR MARKET-BASED BUDGET CONSTRAINT, DEFINED BY THE FACT THAT EXPENDITURE $E(x,y)$ ON x AND y CANNOT EXCEED (AND THUS IS SET EQUAL TO) INCOME OR WEALTH, M:

$$E(x,y) = M$$

$\therefore$ IF x AND y ARE ADJUSTED, BUT SO AS TO REMAIN ON THIS CONSTRAINT, WE HAVE

$$\Delta E = 0; \quad \text{i.e.,} \quad \frac{\partial E}{\partial x}\Delta x + \frac{\partial E}{\partial y}\Delta y \cong 0;$$

$$\text{i.e.,} \quad \frac{\Delta y}{\Delta x} \cong -\frac{\frac{\partial E}{\partial x}}{\frac{\partial E}{\partial y}}; \quad \text{i.e.,} \quad -\frac{\partial y}{\partial x} = \frac{E_x}{E_y}.$$

IF $E(x,y) := p_x x + p_y y$,

THEN THIS BECOMES

$$p_x \Delta x + p_y \Delta y \cong 0 \quad \text{AND} \quad -\frac{\partial y}{\partial x} = \frac{p_x}{p_y}.$$

FOR A MECHANISM DEFINED BY $\pi(m)$ AND $\tau^i(m)$:

$$\Delta y_i \cong -\tau^i_i \Delta m_i \quad \text{AND} \quad \Delta x \cong \pi_i \Delta m_i$$

$$\text{WHERE } \tau^i_i := \frac{\partial \tau^i}{\partial m_i} \quad \text{AND} \quad \pi_i := \frac{\partial \pi}{\partial m_i}.$$

$$\therefore \frac{\Delta y_i}{\Delta x} \cong -\frac{\tau^i_i \Delta m_i}{\pi_i \Delta m_i} = -\frac{\tau^i_i}{\pi_i}$$

$$\text{i.e., } -\frac{\partial y_i}{\partial x} = \frac{\tau^i_i}{\pi_i}$$

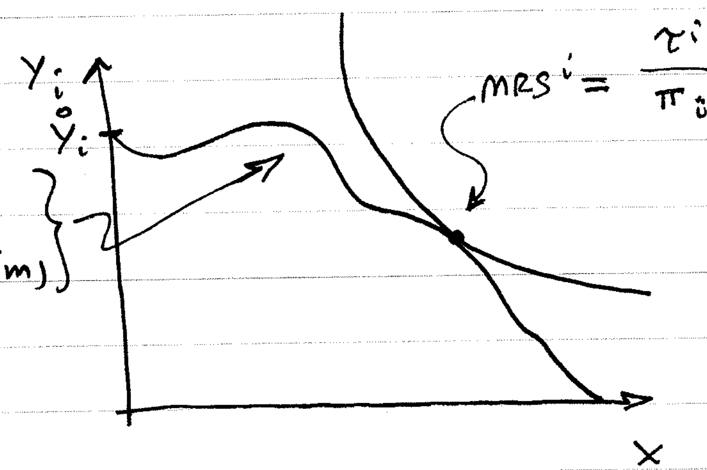
... THE SLOPE OF THE CONSTRAINT ON THE BUNDLE (x, y_i) IS $-\dfrac{\tau^i_i}{\pi_i}$.

EQUIV.: WRITE $y_i = \mathring{y}_i - \tau^i(\pi^{-1}(x))$

$\therefore \dfrac{\partial y_i}{\partial x} = -\tau^i_i \dfrac{1}{\pi_i}$

[π^{-1} IS THE m_i THAT YIELDS x, GIVEN m_{-i}]

$$\left.\begin{aligned} x &= \pi(m) \\ y_i &= \mathring{y}_i - \tau^i(m) \end{aligned}\right\}$$



INDIVIDUAL i'S CHOICE OF m_i WILL THEREFORE SATISFY THE MARGINAL CONDITION

$$\boxed{MRS^i = \frac{\tau^i_i}{\pi_i}.}$$

EXAMPLE 1: GROVES-LEDYARD "QUADRATIC" MECHANISM

$$x = \pi(m) \quad \text{AND} \quad y_i = \overset{\circ}{y}_i - \tau^i(m).$$

$$\pi(m) = \bar{m} = \frac{1}{n}\sum_{j=1}^{n} m_j$$

$$\tau^i(m) = \frac{1}{n}C(\pi(m)) + P^i(m) - R(m),$$

(ξ IS GIVEN)

WHERE $P^i(m) := \xi(m_i - \bar{m})^2$, A "PENALTY"

$R(m) := \frac{1}{n}\sum_1^n P^j(m)$, A "REBATE."

WE HAVE $\pi_i = \frac{1}{n}$ AND

$$\tau^i_i = \frac{1}{n^2}MC + P^i_i - \frac{1}{n}\sum_1^n P^j_i,$$

WHERE $P^i_i = 2\xi(m_i - \bar{m})(1 - \frac{1}{n})$

$P^j_i = 2\xi(m_j - \bar{m})(-\frac{1}{n})$, $j \neq i$.

$$\therefore \tau^i_i = \frac{1}{n^2}MC + \frac{n-1}{n}2\xi(m_i - \bar{m}) +$$

$$+ \frac{1}{n^2}2\xi\underbrace{\sum_{j=1}^{n}(m_j - \bar{m})}_{\equiv 0} - \frac{1}{n}2\xi(m_i - \bar{m})$$

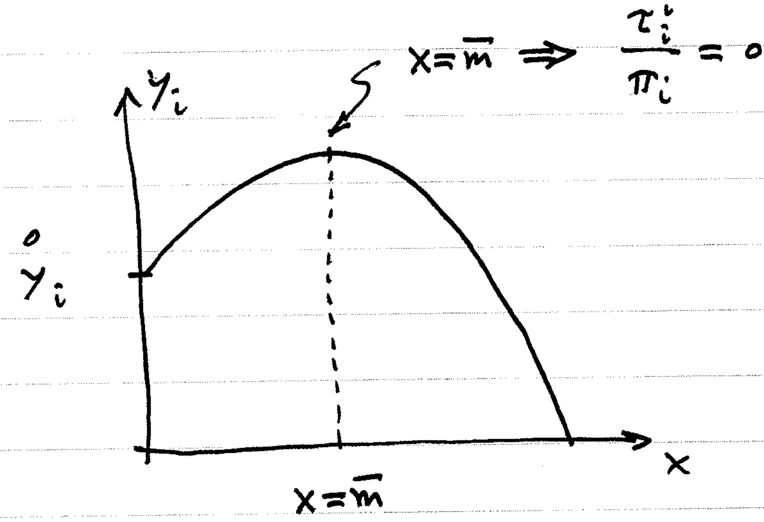
$$= \frac{1}{n^2}MC + 2\xi(m_i - \bar{m})\left(\frac{n-2}{n}\right).$$

$\therefore \frac{\tau^i_i}{\pi_i} = \frac{1}{n}MC + 2\xi(m_i - \bar{m})(n-2)$; THE SLOPE OF THE CONSTRAINT ~~INCREASES~~ LINEARLY AS WE MOVE AWAY FROM $x = \bar{m}$; $\therefore$ THE CONSTRAINT IS QUADRATIC.

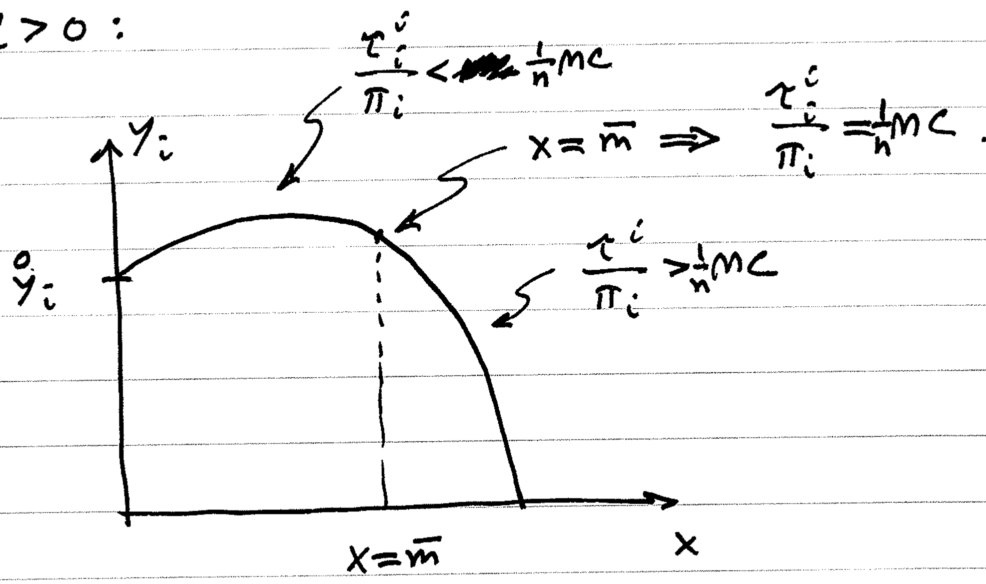
[NOTE: $\sum \frac{\tau^i_i}{\pi_i} \equiv MC$]

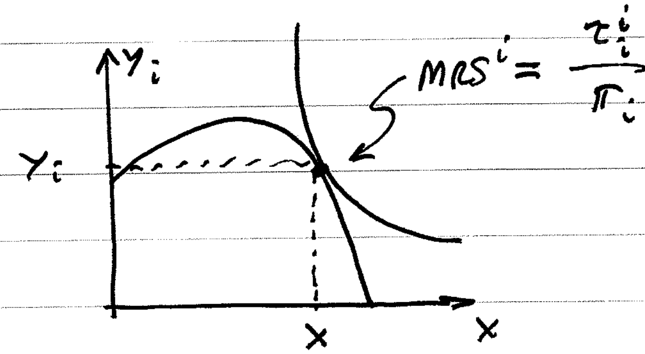
GROVES - LEDYARD

THE CASE $MC = 0$:



IF $MC > 0$:





EXAMPLE 2: WALKER

ASSUME $C(x) = \beta x$. (JUST TO SIMPLIFY)

$\pi(m) = \sum m_j$

$\tau^i(m) = p_i(m)\, x = p_i(m)\, \pi(m)$,

WHERE $p_i(m) = \frac{1}{n}\beta + \xi(m_{i+2} - m_{i+1})$; $\begin{cases} n+1 := 1 \\ n+2 := 2. \end{cases}$

NOTE THAT $p_i(m)$ DOES NOT DEPEND ON m_i:
IF i TAKES $m_{\sim i}$ AS GIVEN, THEN HE TAKES $p_i(m)$ AS GIVEN.

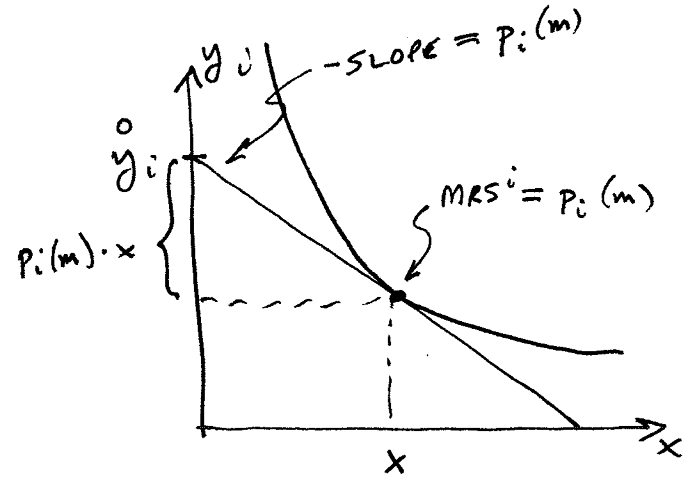
WE HAVE $\tau^i_i := \frac{\partial \tau^i}{\partial m_i} = p_i(m) + \frac{\partial p_i}{\partial m_i} \sum m_j = p_i(m)$. !

$\therefore \frac{\tau^i_i}{\pi_i} = p_i(m)$; $\therefore$ i CHOOSES m_i s.t. $MRS^i = p_i(m)$.

$\therefore \sum MRS^i = \sum p_i(m) = \beta + \xi \sum_1^n (m_{i+1} - m_{i+2}) \equiv \beta = MC$. !

$\therefore$ THE OUTCOME IS PO. BUT ALSO $t_i = (MRS^i) \cdot x$,
SO THIS A <u>LINDAHL EQUILIBRIUM OUTCOME</u>.

BUT WE HAVE REASON TO BE SKEPTICAL OF THE LINDAHL OUTCOME.



THERE IS SOME REASON TO BELIEVE THAT ...
(a) THIS MECHANISM IS UNSTABLE;
(b) PARTICIPANTS WON'T TAKE $m_{\sim i}$ AS "GIVEN."

THIS CALLS INTO QUESTION THE SIMPLE RELIANCE ON NE.