

DESIGNING A MECHANISM WITH PARETO EFFICIENT NASH EQUILIBRIA

(A MECHANISM TO CHOOSE THE PROVISION
LEVEL AND FINANCING OF A PUBLIC GOOD)

THE MECHANISM HAS TO CHOOSE $(x; t_1, \dots, t_n) \in \mathbb{R}_+ \times \mathbb{R}^n$.
LET'S ALLOW PARTICIPANTS TO CHOOSE $m_i \in \mathbb{R}$ ($i=1, \dots, n$).

A CONFIGURATION OF ALL n CHOICES IS AN $m \in \mathbb{R}^n$.

LET'S DENOTE THE OUTCOME FUNCTIONS, OR "RULES," AS:

$x = \pi(m)$, THE "PROVISION FUNCTION"

$y_i = y_i^0 - t_i$, WHERE $t_i = \tau^i(m)$, i 'S "TAX RULE."

IF WE WANT THE NE TO BE PARETO EFFICIENT,
WHAT CONDITIONS WILL THE FUNCTIONS $\pi, \tau^1, \dots, \tau^n$
HAVE TO SATISFY?

WRITE i 'S PAYOFF FUNCTION AS $\tilde{u}^i(m) := u^i(\pi(m), y_i^0 - \tau^i(m))$.

AT AN INTERIOR NE WE MUST HAVE $\frac{\partial \tilde{u}^i}{\partial m_i} = 0$. BUT

$$\frac{\partial \tilde{u}^i}{\partial m_i} = u_x^i \pi_i - u_y^i \tau_i^i, \text{ WHERE } u_x^i = \partial u^i / \partial x, u_y^i = \partial u^i / \partial y,$$

$$\pi_i = \partial \pi / \partial m_i, \tau_i^i = \partial \tau^i / \partial m_i.$$

$$\therefore \frac{\partial \tilde{u}^i}{\partial m_i} = 0 \iff \frac{u_x^i}{u_y^i} = \frac{\tau_i^i}{\pi_i} \text{ i.e., } \boxed{MRS^i = \frac{\tau_i^i}{\pi_i}}$$

SINCE EFFICIENCY REQUIRES $\sum MRS^i = MC$ (IN THE
INTERIOR), WE WILL HAVE TO HAVE

$$\boxed{\sum_{i=1}^n \frac{\tau_i^i}{\pi_i} = MC.}$$

IF THE OTHERS' ACTIONS (i.e., THE $(n-1)$ -TUPLE m_{-i}) ARE TAKEN AS GIVEN, THEN THE CHOICE OF m_i BY INDIVIDUAL i IMPLIES A CHOICE OF BOTH x AND y_i . OF COURSE, HIS CHOICE IS CONSTRAINED TO SATISFY SOME KIND OF TRADE-OFF BETWEEN x AND y_i — i.e., THE SET OF (x, y_i) PAIRS AVAILABLE BY VARYING THE CHOICE m_i IS CONSTRAINED, AND THE TRADE-OFF IS THE SLOPE OF THAT CONSTRAINT.

THE PROTOTYPE FOR THIS IDEA IS THE FAMILIAR MARKET-BASED BUDGET CONSTRAINT, DEFINED BY THE FACT THAT EXPENDITURE $E(x, y)$ ON x AND y CANNOT EXCEED (AND THUS IS SET EQUAL TO) INCOME OR WEALTH, M :

$$E(x, y) = M$$

∴ IF x AND y ARE ADJUSTED, BUT SO AS TO REMAIN ON THIS CONSTRAINT, WE HAVE

$$\Delta E = 0; \text{ i.e., } \frac{\partial E}{\partial x} \Delta x + \frac{\partial E}{\partial y} \Delta y \approx 0;$$

$$\text{i.e., } \frac{\Delta y}{\Delta x} \approx - \frac{\frac{\partial E}{\partial x}}{\frac{\partial E}{\partial y}}; \text{ i.e., } - \frac{\partial y}{\partial x} = \frac{E_x}{E_y}.$$

IF $E(x, y) := p_x x + p_y y$, THEN THIS BECOMES

$$p_x \Delta x + p_y \Delta y \approx 0 \quad \text{AND} \quad - \frac{\partial y}{\partial x} = \frac{p_x}{p_y}.$$

FOR A MECHANISM DEFINED BY $\pi(m)$ AND $\tau^i(m)$:

$$\Delta y_i \approx -\tau_i^i \Delta m_i \quad \text{AND} \quad \Delta x \approx \pi_i \Delta m_i$$

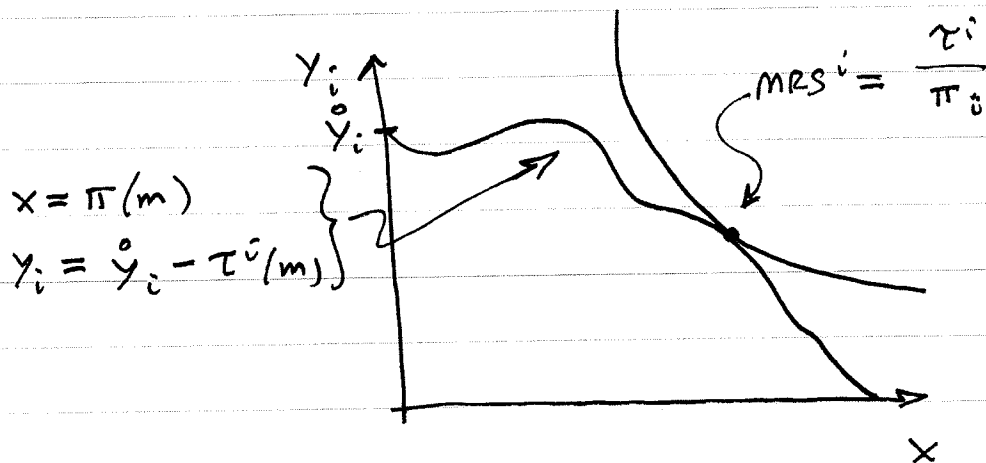
$$\text{WHERE } \tau_i^i := \frac{\partial \tau^i}{\partial m_i} \quad \text{AND} \quad \pi_i := \frac{\partial \pi}{\partial m_i}.$$

$$\therefore \frac{\Delta y_i}{\Delta x} \approx - \frac{\tau_i^i \Delta m_i}{\pi_i \Delta m_i} = - \frac{\tau_i^i}{\pi_i}$$

$$\text{i.e., } - \frac{\partial y_i}{\partial x} = \frac{\tau_i^i}{\pi_i}$$

... THE SLOPE OF THE CONSTRAINT ON THE BUNDLE (x, y_i) IS $-\frac{\tau_i^i}{\pi_i}$.

EQUIV.:
 WRITE
 $y_i = y_i^0 - \tau^i(\pi^i(x))$
 $\therefore \frac{\partial y_i}{\partial x} = -\tau_i^i \frac{1}{\pi_i}$
 [π^i IS THE m_i THAT YIELDS x , GIVEN m_{-i}]



INDIVIDUAL i 'S CHOICE OF m_i WILL THEREFORE SATISFY THE MARGINAL CONDITION

$$\boxed{MRS^i = \frac{\tau_i^i}{\pi_i}}$$

EXAMPLE 1: GROVES-LEDYARD "QUADRATIC" MECHANISM

$$X = \pi(m) \quad \text{AND} \quad y_i = \bar{y}_i - \tau^i(m).$$

$$\pi(m) = \bar{m} = \frac{1}{n} \sum_{j=1}^n m_j$$

$$\tau^i(m) = \frac{1}{n} C(\pi(m)) + P^i(m) - R(m), \quad \left(\frac{C}{\pi} \text{ IS GIVEN} \right)$$

WHERE $P^i(m) := \frac{2}{3} (m_i - \bar{m})^2$, A "PENALTY"

$R(m) := \frac{1}{n} \sum_{j=1}^n P^j(m)$, A "REBATE."

WE HAVE $\pi_i = \frac{1}{n}$ AND

$$\tau_i^i = \frac{1}{n^2} MC + P_i^i - \frac{1}{n} \sum_{j=1}^n P_i^j,$$

WHERE $P_i^i = 2 \frac{2}{3} (m_i - \bar{m}) (1 - \frac{1}{n})$

$P_i^j = 2 \frac{2}{3} (m_j - \bar{m}) (-\frac{1}{n})$, $j \neq i$.

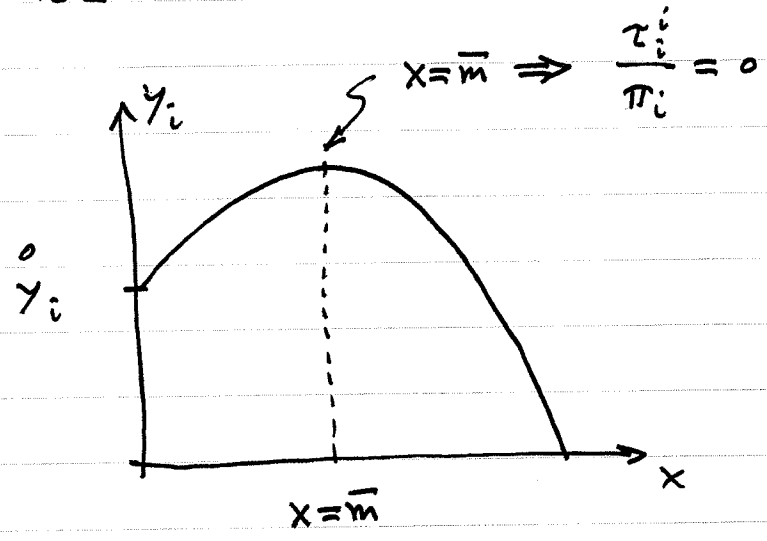
$$\begin{aligned} \therefore \tau_i^i &= \frac{1}{n^2} MC + \frac{n-1}{n} 2 \frac{2}{3} (m_i - \bar{m}) + \\ &\quad + \frac{1}{n^2} 2 \frac{2}{3} \underbrace{\sum_{j=1}^n (m_j - \bar{m})}_{\equiv 0} - \frac{1}{n} 2 \frac{2}{3} (m_i - \bar{m}) \end{aligned}$$

$$= \frac{1}{n^2} MC + 2 \frac{2}{3} (m_i - \bar{m}) \left(\frac{n-2}{n} \right).$$

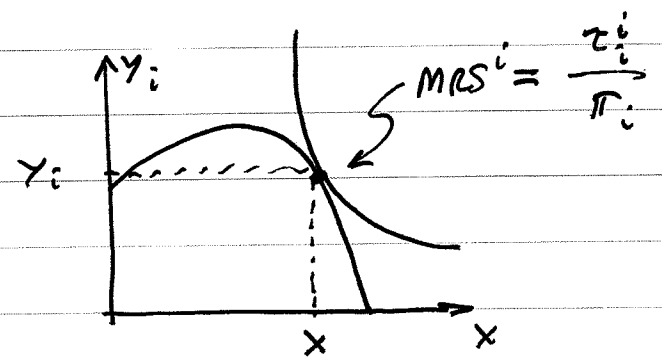
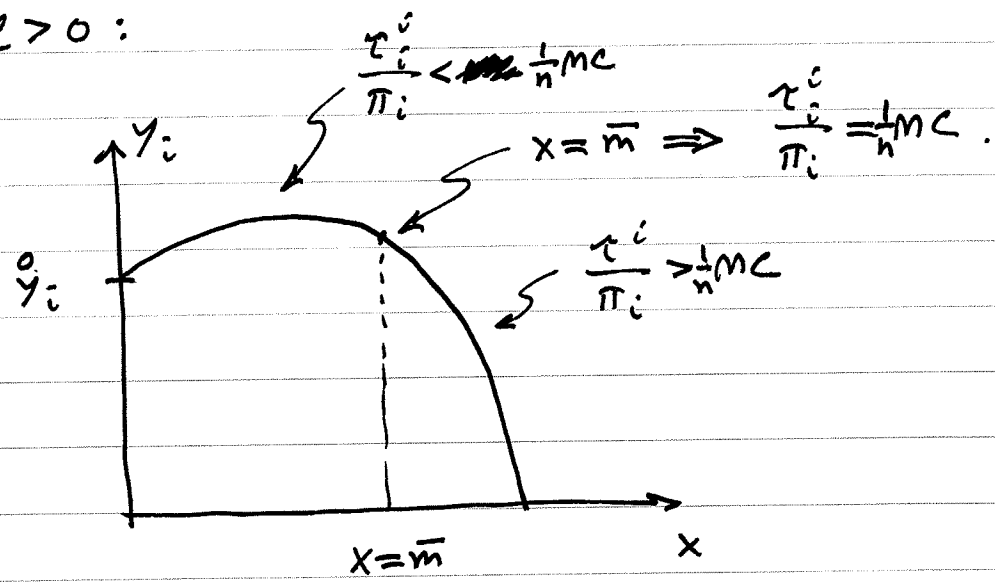
$\therefore \frac{\tau_i^i}{\pi_i} = \frac{1}{n} MC + 2 \frac{2}{3} (m_i - \bar{m})^{(n-2)}$; THE SLOPE OF THE
CONSTRAINT INCREASES LINEARLY AS WE
MOVE AWAY FROM $X = \bar{m}$; \therefore THE

CONSTRAINT IS QUADRATIC. [NOTE: $\sum \frac{\tau_i^i}{\pi_i} \equiv MC$]

THE CASE $MC=0$:



IF $MC > 0$:



EXAMPLE 2:

WALKER

ASSUME $C(x) = \beta x$.
 (JUST TO SIMPLIFY)

$\pi(m) = \sum m_j$

$\tau^i(m) = p_i(m) x = p_i(m) \pi(m)$,

WHERE $p_i(m) = \frac{1}{n} \beta + \xi (m_{i+2} - m_{i+1})$; $\begin{cases} n+1 := 1 \\ n+2 := 2 \end{cases}$.

NOTE THAT $p_i(m)$ DOES NOT DEPEND ON m_i :

IF i TAKES m_{-i} AS GIVEN, THEN HE TAKES $p_i(m)$ AS GIVEN.

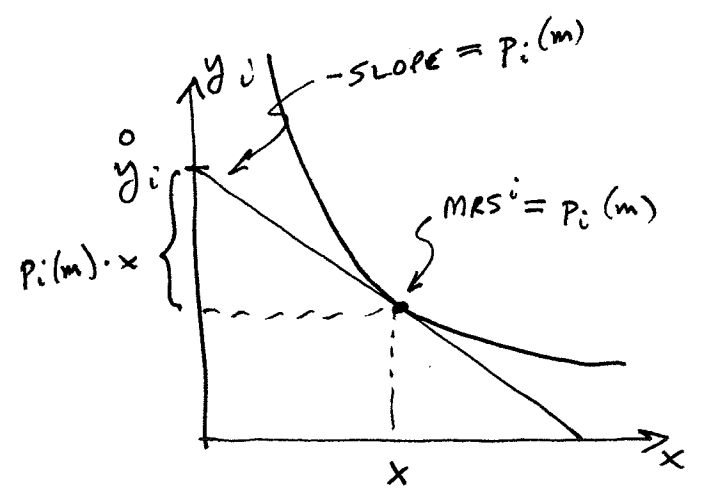
WE HAVE $\tau_i^i := \frac{\partial \tau^i}{\partial m_i} = p_i(m) + \frac{\partial p_i}{\partial m_i} \sum m_j = p_i(m)$. !

$\therefore \frac{\tau_i^i}{\pi_i} = p_i(m)$; $\therefore i$ CHOOSES m_i S.T. $MRS^i = p_i(m)$.

$\therefore \sum MRS^i = \sum p_i(m) = \beta + \xi \sum (m_{i+1} - m_{i+2}) \equiv \beta = MC$. !

\therefore THE OUTCOME IS PO. BUT ALSO $t_i = (MRS^i) \cdot x$,
 SO THIS A LINDAHL EQUILIBRIUM OUTCOME.

BUT WE HAVE REASON TO BE SKEPTICAL OF THE LINDAHL OUTCOME.

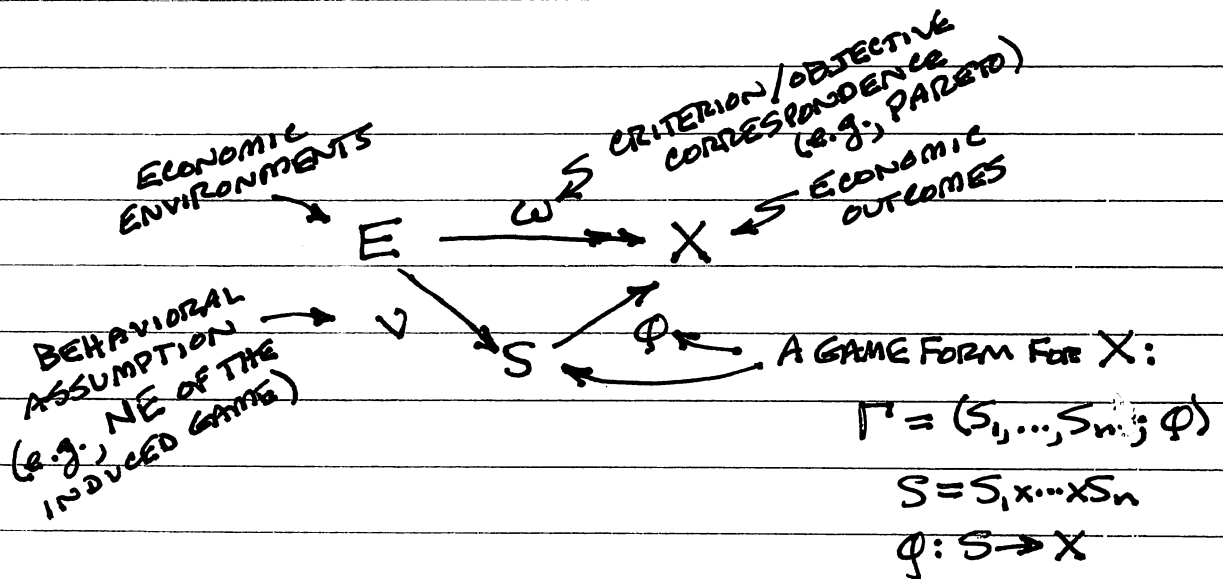


THERE IS SOME REASON TO BELIEVE THAT ...

- (a) THIS MECHANISM IS UNSTABLE;
- (b) PARTICIPANTS WON'T TAKE m_{-i} AS "GIVEN."

THIS CALLS INTO QUESTION THE SIMPLE RELIANCE ON NE.

MECHANISM DESIGN: THE REITER DIAGRAM



THE SETS E AND X AND THE CORRESPONDENCE ω ARE GIVEN, AND EACH $e \in E$ INCLUDES (AT LEAST) UTILITY FUNCTIONS $u^i: X \rightarrow \mathbb{R}$ ($i=1, \dots, n$).

THE FUNCTION (OR CORRESPONDENCE) v DESCRIBES OUR ASSUMPTION ABOUT WHAT CHOICES $s_i \in S_i$ THE PARTICIPANTS WILL MAKE IN EACH $e \in E$.

THE GAME FORM $\Gamma = (S_1, \dots, S_n; \phi)$ IS THE "VARIABLE" IN THE MECHANISM DESIGN PROBLEM, WHICH CAN BE CHOSEN. WE WISH TO CHOOSE A Γ THAT SATISFIES $\phi(v(e)) = \omega(e)$ OR $\phi(v(e)) \in \omega(e)$ OR $\phi(v(e)) \subseteq \omega(e)$ FOR EACH $e \in E$.

IN OUR PUBLIC GOOD PROBLEM:

$x \in X$: $x = (x, y_1, \dots, y_n) \in \mathbb{R}_+^{n+1}$.

$e \in E$: $e = (u^i)_1^n$, OR $e = (u^i, y_i)_1^n$.

$S \in S$: WE DECIDED TO SAY $s_i = m_i \in \mathbb{R}$ OR \mathbb{R}_+ ; $S_i = \mathbb{R}$ OR \mathbb{R}_+ .

ϕ : WE DERIVED CONDITIONS ϕ WOULD HAVE TO SATISFY TO ENSURE $\forall e \in E$: $\phi(v(e))$ IS PARETO FOR e .