NONCOOPERATIVE GAMES AND NASH EQUILIBRIUM

FOR ANY POSITIVE INTEGER n, AN n-PLAYER NONCOOPERATIVE NORMAL FORM GAME (OR JUST AN n-PLAYER GAME) CONSISTS OF

(1) n STRATEGY SETS $S_1, \dots, S_n$,

AND (2) n PAYOFF FUNCTIONS $\pi_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ $(i=1,\dots,n)$;

i.e., A GAME IS AN n-TUPLE $(S_i, \pi_i)_1^n$, WHERE $\pi_i : \overset{n}{\underset{i=1}{\times}} S_i \to \mathbb{R}$ FOR EACH i.

A USEFUL ALTERNATIVE FORMULATION:

DEFN: A GAME FORM CONSISTS OF

(1) n STRATEGY SETS $S_1, \dots, S_n$

AND (2) AN OUTCOME FUNCTION $\varphi : S_1 \times \cdots \times S_n \to X$,

WHERE X IS A SET OF POSSIBLE OUTCOMES.

REMARK: GIVEN A GAME FORM $(S_1, \dots, S_n; \varphi)$ AND n UTILITY FUNCTIONS $u_i : X \to \mathbb{R}$ $(i=1,\dots,n)$, WE CAN DEFINE n PAYOFF FUNCTIONS

$$\pi_i(s_1, \dots, s_n) := u_i(\varphi(s_1, \dots, s_n)), \quad i=1,\dots,n,$$

AND THE RESULTING n-TUPLE $(S_i, \pi_i)_1^n$ IS AN n-PLAYER GAME.

NASH EQUILIBRIUM:

DEFN: A NASH EQUILIBRIUM OF THE n-PLAYER GAME $(S_i, \pi_i)_1^n$ IS AN n-TUPLE $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_n) \in \overset{n}{\underset{i=1}{\times}} S_i$ THAT SATISFIES

$$\forall i: \forall s_i \in S_i: \pi_i(\hat{s}) \geqq \pi_i(s_i, \hat{s}_{\sim i}),$$

WHERE $(s_i, \hat{s}_{\sim i}) := (\hat{s}_1, \ldots, \hat{s}_{i-1}, s_i, \hat{s}_{i+1}, \ldots, \hat{s}_n)$.

EXAMPLE 1: (COURNOT EQUILIBRIUM)

$s_i \in S_i$ IS THE QUANTITY CHOICE $q_i \in \mathbb{R}_+$;

THE PAYOFF FUNCTION π_i IS FIRM i'S PROFIT FUNCTION (WHICH WE ALSO DENOTED Π_i).

EXAMPLE 2: (BERTRAND EQUILIBRIUM)

$s_i \in S_i$ IS FIRM i'S PRICE $p_i \in \mathbb{R}_+$;

THE PAYOFF FUNCTION π_i IS FIRM i'S PROFIT FUNCTION $\Pi_i(p_1, p_2, \ldots, p_n)$.

A PLAYER'S REACTION FUNCTION

(OR BEST-RESPONSE FUNCTION)

DEFN: IN AN n-PLAYER GAME $(S_i, \pi_i)_1^n$, PLAYER i'S REACTION CORRESPONDENCE IS DEFINED BY

$$r_i(s_{\sim i}) := \{ s_i \in S_i \mid \forall s_i' \in S_i : \pi_i(s_i, s_{\sim i}) \geq \pi_i(s_i', s_{\sim i}) \}.$$

IN OTHER WORDS, FOR ANY CONFIGURATION $s_{\sim i}$ OF CHOICES BY THE OTHER $n-1$ PLAYERS, $r_i(s_{\sim i})$ IS THE SET OF ALL CHOICES s_i BY PLAYER i THAT ARE BEST CHOICES AGAINST $s_{\sim i}$ — i.e., CHOICES FOR i THAT WOULD BE "BEST RESPONSES" IF i KNEW THAT THE OTHERS WOULD CHOOSE $s_{\sim i}$.

NOTE THAT $r_i(\cdot)$ WILL GENERALLY BE A CORRESPONDENCE (i.e., WILL NOT BE SINGLE-VALUED), AND THAT IF π_i IS NOT CONTINUOUS OR IF STRATEGY SPACES ARE NOT COMPACT, THEN $r_i(s_{\sim i})$ MAY BE UNDEFINED FOR SOME $s_{\sim i}$.

SOME SIMPLE TWO-PLAYER GAMES

① "APPROACHING ONE ANOTHER ON A ROAD"

EAST

		L	R
WEST	L	0, 0	−1, −1
	R	−1, −1	0, 0

TWO EQUILIBRIA: (L, L) AND (R, R).
(IN PURE STRATEGIES)

② "MATCHING PENNIES"

COLUMN

		H	T
ROW	H	−1, 1	1, −1
	T	1, −1	−1, 1

NO EQUILIBRIUM (IN PURE STRATEGIES).

③ "PAPER, SCISSORS, ROCK"

COLUMN

		P	S	R
	P	0, 0	−1, 1	1, −1
ROW	S	1, −1	0, 0	−1, 1
	R	−1, 1	1, −1	0, 0

NO EQUILIBRIUM (IN PURE STRATEGIES).

④

	L	M	R
T	1, 0	−1, 0	0, −1
B	1, 0	2, −1	−1, 1

ONLY EQUIL'M IS (T, L), BUT T IS NOT UNIQUELY BEST AGAINST L, NOR L AGAINST T.

⑤

	L	R
T	1, 1	2, 0
B	0, 2	1, 1

ONLY EQUIL'M IS (T, L). BOTH T AND L ARE DOMINANT STRATEGIES.

REACTION FUNCTIONS IN THE EXAMPLES:

(1)

s_2	L	R
$r_1(s_2)$	L	R

s_1	L	R
$r_2(s_1)$	L	R

(2)

s_2	H	T
$r_1(s_2)$	T	H

s_1	H	T
$r_2(s_1)$	H	T

(3)

s_2	P	S	R
$r_1(s_2)$	S	R	P

s_1	P	S	R
$r_2(s_1)$	S	R	P

(4)

s_2	L	M	R
$r_1(s_2)$	$\{T, B\}$	B	T

s_1	T	B
$r_2(s_1)$	$\{L, M\}$	R

(5)

s_2	L	R
$r_1(s_2)$	T	T

s_1	T	B
$r_2(s_1)$	L	L