The Cournot and Bertrand Models
of Industry Equilibrium

Now we're going to remove the assumption of price-taking behavior by firms. We'll begin with the elementary theory of the firm, and then we'll apply the theory to the case of a monopoly. Then we'll move on to strategic behavior and equilibrium when there are multiple firms in a market.

The Elementary Theory of the Firm

In the most basic theory of the firm, a single-product firm chooses its level of production to maximize its profit, which is the difference between its revenue and its cost. We model its decision problem as

$$\max_{q \in \mathbb{R}_+} \pi(q) := R(q) - C(q).$$

Assuming that the revenue and cost functions are twice differentiable, the first-order marginal condition is

$$\pi'(q) = 0 \;\text{ if } q > 0 \qquad \text{and} \qquad \pi'(q) \leqq 0 \;\text{ if } q = 0,$$

i.e.,

$$R'(q) = C'(q) \;\text{ if } q > 0 \qquad \text{and} \qquad R'(q) \leqq C'(q) \;\text{ if } q = 0,$$

i.e.,

$$MR(q) = MC(q) \;\text{ if } q > 0 \qquad \text{and} \qquad MR(q) \leqq MC(q) \;\text{ if } q = 0.$$

The necessary second-order condition is that $\pi''(q) \leqq 0$, *i.e.*, that

$$MR'(q) \leqq MC'(q), \qquad \textit{i.e.}\text{, that "MC cuts MR from below".}$$

The second-order condition would be satisfied, for example, if

$$MR \text{ is decreasing} \qquad \text{and} \qquad MC \text{ is increasing.}$$

So far, this is just reprising what we all learned in our principles of economics course, where we probably saw a diagram like Figure 1.

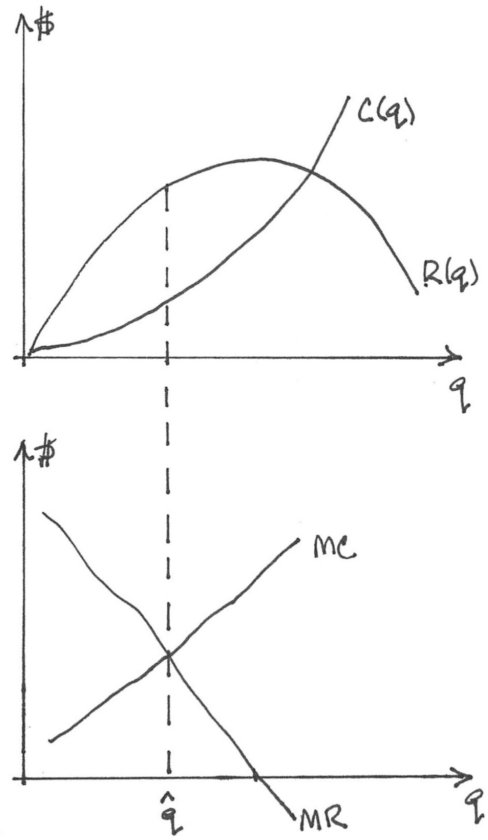

Figure 1

A Monopoly Example

Let's apply the basic theory of the firm to a simple numerical example of a monopoly. The market demand function for the firm's product, and the firm's cost function, are as follows:

Market demand: $Q = D(p) = 50 - \frac{1}{2}p$; the inverse demand function is $p = 100 - 2Q$.

Cost function: $C(Q) = 40Q$.

The firm's revenue function is $R(Q) = (100 - 2Q)Q = 100Q - 2Q^2$, so we have

$$MR = 100 - 4Q \qquad \text{and} \qquad MC = 40,$$

Our $MR = MC$ first-order condition yields $Q = 15$ and $p = \$70$. The firm's profit is $\pi = \$1050 - \$600 = \$450$; consumer surplus is \$225; and total surplus is \$675. See Figure 2.

Note that we could just as well have used the *price* as the firm's decision variable, and the solution (the firm's decision) would have been the same:

The revenue function: $\widetilde{R}(p) = pQ = (50 - \frac{1}{2}p)p = 50p - \frac{1}{2}p^2$.

The cost function: $\widetilde{C}(p) = 40Q = 40(50 - \frac{1}{2}p) = 2000 - 20p$.

The profit function:
$$\begin{aligned}\widetilde{\pi}(p) &= \widetilde{R}(p) - \widetilde{C}(p) \\ &= (50p - \tfrac{1}{2}p^2) - (2000 - 20p) \\ &= 70p - \tfrac{1}{2}p^2 - 2000.\end{aligned}$$

The derivative of the profit function is $\widetilde{\pi}'(p) = 70 - p$, so the first-order condition $\widetilde{\pi}'(p) = 0$ yields $p = \$70$ and $Q = 15$, just as before. This of course follows from the fact that, given the market demand curve, choosing the selling price determines the quantity that will be sold, and choosing the quantity to be sold determines the market-clearing price.

Note too that nothing here depended on the firm actually being a monopoly. All that mattered was that the demand curve for the firm's product is downward-sloping: if the firm increases the price it charges, it will lose some sales, but if the increase is not too large then the firm won't lose *all* its sales. We say that a firm has some "market power" when it faces a downward-sloping demand for its product (instead of a horizontal demand curve, where raising its price will cause it to lose all its sales — *i.e.*, where it has to take the price as given). Therefore this "monopoly" decision model is actually a model that applies to the profit-maximizing decision of any firm with market power — any firm that's not a price-taker.

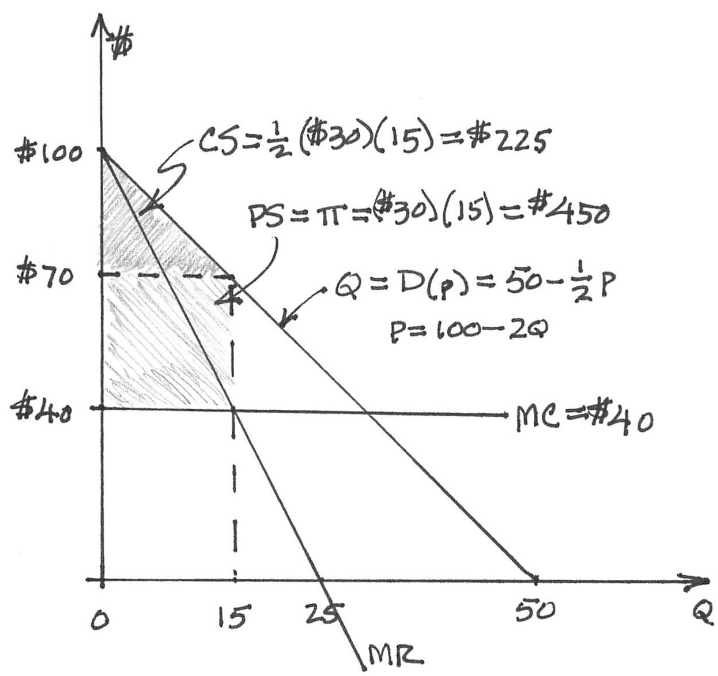

Figure 2

The Cournot Model and Cournot Equilibrium

Now let's assume there are two firms in the market. In order to compare this situation to the monopoly we just analyzed, let's suppose a new firm has entered the monopolist's market and the new firm is identical to the original firm: the new firm produces exactly the same product as the first firm — so consumers make no distinction between the two firms' products — and the new firm has the same cost function as the first firm.

Therefore the market demand function and the firms' cost functions are as follows, where q_1 and q_2 denote the output quantities chosen by the two firms:

Market demand: $p = 100 - 2Q$, where $Q = q_1 + q_2$ is the total quantity sold.

Cost functions: $C_1(q_1) = 40q_1$ and $C_2(q_2) = 40q_2$.

When we analyzed the monopolist, we assumed that the firm takes the demand function for its product as given and chooses the output quantity that maximizes its profit. Now, with two firms in the market, in order for either firm to have a well-defined decision problem we'll have to assume that in addition to each firm taking the market demand function as given, each firm will also have to make some assumption about what its rival firm is going to do.

Cournot Behavioral Assumption: We assume that each firm chooses its output quantity q_i to maximize its profit, taking its rival's output (and the market demand function) as given.

Let's consider Firm 2's decision problem:

$$\begin{aligned} \max_{q_2 \in \mathbb{R}_+} \pi_2(q_1, q_2) &= R_2(q_1, q_2) - C_2(q_2) \\ &= [100 - 2(q_1 + q_2)]q_2 - 40q_2 \\ &= [100 - 2q_1]q_2 - 2q_2^2 - 40q_2. \end{aligned}$$

The first-order marginal condition for this problem is

$$\frac{\partial \pi_2}{\partial q_2} = 0 \text{ if } q_2 > 0 \quad \text{and} \quad \frac{\partial \pi_2}{\partial q_2} \leqq 0 \text{ if } q_2 = 0.$$

In our numerical example this first-order condition is

$$\begin{aligned} 100 - 2q_1 - 4q_2 &= 40 \text{ if } q_2 > 0 \qquad (1) \\ 100 - 2q_1 - 4q_2 &\leqq 40 \text{ if } q_2 = 0, \end{aligned}$$

and the solution function for Firm 2's decision problem is therefore

$$q_2 = r_2(q_1) = \begin{cases} 15 - \frac{1}{2}q_1, & \text{if } q_1 \leqq 30 \\ 0, & \text{if } q_1 \geqq 30, \end{cases}$$

which is Firm 2's **reaction function**. The reaction function is depicted in Figure 3.

Note that the first-order condition (1) is actually the familiar $MR = MC$ (if $q_2 > 0$) and $MR \leqq MC$ (if $q_2 = 0$). This suggests that the firm's decision problem is exactly like the monopolist's problem, except that here the firm's marginal revenue function has to include the output chosen by Firm 1, which is treated as a parameter by Firm 2. We'll return to this idea below.

Turning to Firm 1's decision problem, we note that because the firms are identical, Firm 1's problem is exactly the same as Firm 2's problem with the subscripts reversed. Therefore Firm 1's first-order condition in our numerical example is

$$\begin{aligned} 100 - 4q_1 - 2q_2 &= 40 \text{ if } q_1 > 0 \\ 100 - 4q_1 - 2q_2 &\leqq 40 \text{ if } q_1 = 0, \end{aligned} \tag{2}$$

and the solution function for Firm 1's decision problem is

$$q_1 = r_1(q_2) = \begin{cases} 15 - \frac{1}{2}q_2, & \text{if } q_2 \leqq 30 \\ 0, & \text{if } q_2 \geqq 30, \end{cases}$$

which is Firm 1's reaction function.

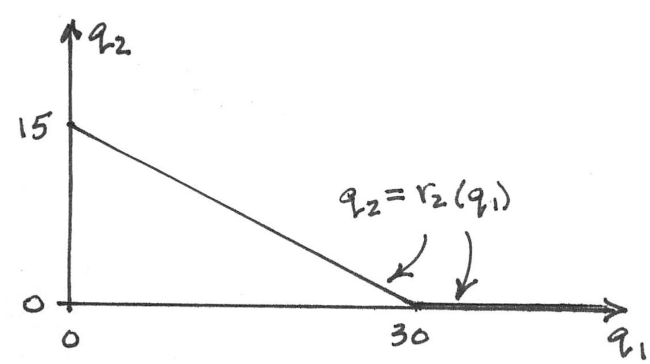

Figure 3

Cournot Equilibrium

If the firms behave according to the Cournot assumption, then the natural definition of equilibrium is a pair of output choices (q_1, q_2) in which each firm is taking the other's choice as given:

Definition: A **Cournot equilibrium** in a market with two firms is a pair of quantity choices, $(\widehat{q}_1, \widehat{q}_2) \in \mathbb{R}^n_+$, that satisfies the condition

$$\widehat{q}_1 \text{ maximizes } \pi_1(q_1, \widehat{q}_2) \quad \text{and} \quad \widehat{q}_2 \text{ maximizes } \pi_2(\widehat{q}_1, q_2).$$

In other words, an equilibrium is a pair of quantity choices that satisfy both firms' reaction functions:

$$\widehat{q}_1 = r_1(\widehat{q}_2) \quad \text{and} \quad \widehat{q}_2 = r_2(\widehat{q}_1).$$

In our numerical example, then, an equilibrium is a simultaneous solution of the two firms' first-order equations in (1) and (2),

$$4q_1 + 2q_2 = 60 \quad \text{and} \quad 2q_1 + 4q_2 = 60,$$

if both q_1 and q_2 are positive. There is clearly a unique solution in the example, *i.e.*, a unique Cournot equilibrium: $(\widehat{q}_1, \widehat{q}_2) = (10, 10)$, at which the price is $p = \$60$ and each firm's profit is $\pi_i = \$200$. Consumer surplus is $CS = \frac{1}{2}(\$40)(20) = \$400$ and total surplus is \$800. Figure 4 depicts both firms' reaction functions and the Cournot equilibrium.

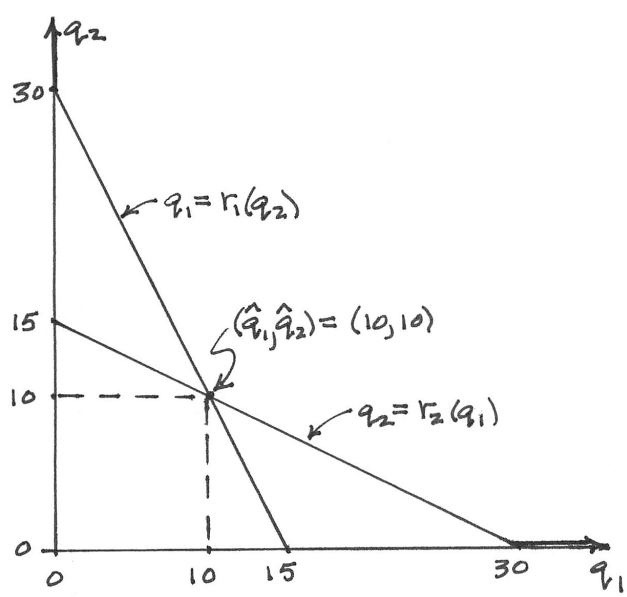

Figure 4

Residual Demand

Before generalizing to non-identical firms, to more than two firms, etc., let's first revisit the firm's decision problem, looking at the problem from the perspective of the elementary theory of the firm. We'll focus on Firm 1's decision problem, in a way that will make it easy to compare the decision problem when there are two firms to the decision problem when the firm was a monopoly.

Here's the market demand function in our numerical example:

$$Q = D(p) = 50 - \tfrac{1}{2}p.$$

We're assuming that Firm 1 knows the market demand curve, which it takes as given, and we're also assuming that Firm 1 takes q_2 as given (the Cournot behavioral assumption). Therefore, from Firm 1's perspective, the demand for its output can be expressed as

$$q_1 = D(p) - q_2 = 50 - \tfrac{1}{2}p - q_2 = [50 - q_2] - \tfrac{1}{2}p.$$

That's the **residual demand function** Firm 1 faces for its product — the demand for Firm 1's output that remains, at any price p, if Firm 2 is supplying q_2 units to the market. Let's denote this residual demand function, for a given value of q_2, as $D_1(p; q_2) := D(p) - q_2$.

The inverse demand function defined by the residual demand in our example is

$$p = 100 - 2Q = 100 - 2q_1 - 2q_2 = [100 - 2q_2] - 2q_1,$$

and Firm 1 is taking q_2, and therefore the entire term in the brackets, as given. Therefore this market inverse demand function has exactly the same form as the linear market inverse demand function Firm 1 faced when it was a monopoly, except that the vertical-axis intercept term is now $100-2q_2$ instead of 100, and the horizontal-axis intercept is now $50-q_2$ instead of just 50. So the analysis we applied to the monopolist should apply in the same way here, with a (residual) demand curve that's simply the market demand curve shifted horizontally to the left by q_2 units, as in Figure 5. Just as we indicated earlier, the model of the monopolist's decision is actually the right model for any firm's decision — if we replace the market demand curve by the residual demand curve the firm faces.

Indeed, we now have

$$MR = [100 - 2q_2] - 4q_1 \qquad \text{and} \qquad MC = 40,$$

so the $MR = MC$ first-order condition yields $4q_1 = 60 - 2q_2$, *i.e.*, $q_1 = 15 - \tfrac{1}{2}q_2$, the same solution function (reaction function) we obtained earlier.

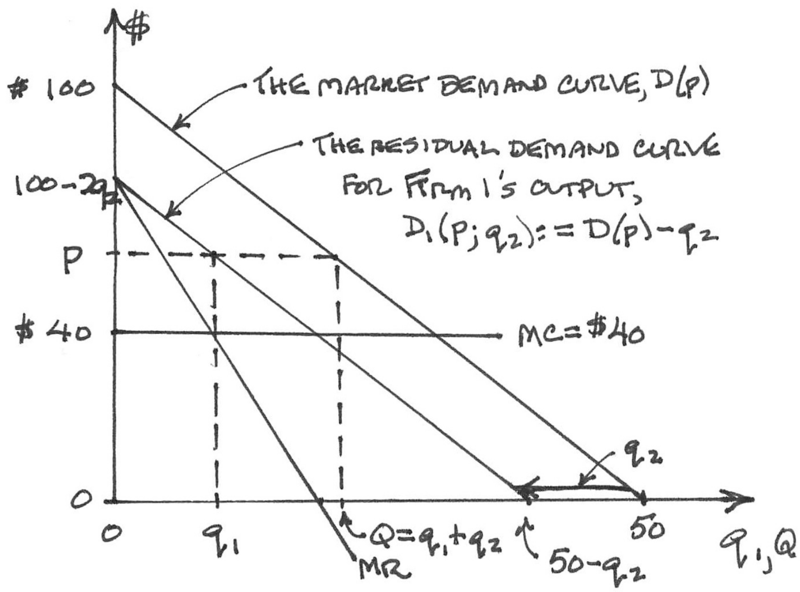

Figure 5

Generalizing the Example

Let's see how things change if we consider a more general situation than the one in our example. For each of the generalizations we're going to consider, there are examples in the Exercise Book.

First, suppose the firms have **asymmetric costs**, *i.e.*, they don't have identical cost functions. Then the reaction functions will not be mirror images of one another as they were in Figure 4, and the equilibrium will not be symmetric, *i.e.*, we won't generally have $q_1 = q_2$. Moreover, now there may be boundary equilibria, in which one of the firms chooses $q_i = 0$.

Next, what if the firms are producing **differentiated products**, products that aren't identical. In this case the firms will typically be able to sell their products at prices that need not be the same. If the demand functions for their products are linear, for example, then the inverse demand functions would have the form

$$p_1 = b_1 - a_{11}q_1 - a_{12}q_2 \qquad \text{and} \qquad p_2 = b_2 - a_{21}q_1 - b_{22}q_2. \tag{3}$$

These yield quadratic revenue functions and therefore linear marginal revenue functions, so (if marginal cost functions are linear) we would continue to have linear reaction functions, as in Figures 3 and 4. We still have residual demand functions $D_1(p_1; q_2)$ and $D_2(p_2; q_2)$, and inverse residual demand functions (in (3) above, with the rival firm's q_i taken as given), although this is a slight abuse of the word "residual," since the total quantity $Q = q_1 + q_2$ never enters the analysis if the firms' products are differentiated.

What if we have **nonlinear demand or cost functions**? The principles are the same, but we may not be able to obtain closed-form solutions to the two equilibrium equations so easily, if at all.

And what if there are **more than two firms**? Again, the principles are the same, but now we'll have as many equilibrium conditions as there are firms, according to the following general definition of Cournot equilibrium:

Definition: A **Cournot equilibrium** in a market with n firms is a profile of quantity choices, $(\widehat{q}_1, \ldots, \widehat{q}_n) \in \mathbb{R}^n_+$, that satisfies the condition

$$\forall i \in \{1, \ldots, n\} : \ \widehat{q}_i \text{ maximizes } \pi_i(q_i, \widehat{\mathbf{q}}_{-i}),$$

where $(q_i, \widehat{\mathbf{q}}_{-i})$ is the n-tuple $(\widehat{q}_1, \ldots, \widehat{q}_{i-1}, q_i, \widehat{q}_{i+1}, \ldots, \widehat{q}_n)$.

The equilibrium conditions will be the first-order conditions for each of the n firms' maximization problems, a system of n equations or inequalities in the n decision variables $q_1, \ldots, q_n$.

The Bertrand Model and Bertrand Equilibrium

The Cournot model is often referred to as one in which the firms choose quantities, or, it's said, they "compete in quantities." This is a very misleading terminology, as we're going to see, but let's stay with it for a while. An obvious alternative model is one in which the firms choose prices, or "compete in prices" — the **Bertrand model** of firms behaving strategically.

Ideally, we would continue to use the same numerical example for introducing the Bertrand model as we used in developing the Cournot model. However that was a very special case: we assumed the firms' products were identical, and consumers were therefore indifferent about which firm they purchased from. Each firm's output would therefore sell at the same price. In the Cournot model this didn't pose a problem: we assumed the firms chose output quantities, and the price that resulted was whatever price cleared the market. But if the firms are choosing prices, and if they choose *different* prices, then the firm charging the lower price would presumably garner all, or nearly all, the sales, and the high-price firm would sell few if any units. In the Bertrand model this is a very special case; the analysis is completely different in the Bertrand model if the firms produce differentiated products (unlike the Cournot model, where the analysis is the same whether the firms' products are homogeneous or differentiated). For the Bertrand model we'll first analyze the case of differentiated products, followed by the homogeneous-products case.

Let's begin, then, with a different numerical example, still with only two firms. We'll assume the demand functions for the firms' products are

$$q_1 = D_1(p_1, p_2) = 30 - \tfrac{2}{3}p_1 + \tfrac{1}{3}p_2 \quad \text{and} \quad q_2 = D_2(p_1, p_2) = 30 - \tfrac{2}{3}p_2 + \tfrac{1}{3}p_1$$

when $p_1, p_2 \leqq 90$. We'll also assume that the firms have no costs: for a concrete example, think of fish that each firm already has on hand and which will spoil if not sold. Firm 1 sells salmon, let's say, and Firm 2 sells sea bass (so their products are differentiated). Since costs are identically zero, a firm's revenue is its profit. We replace the Cournot behavioral assumption with the Bertrand behavioral assumption:

Bertrand Behavioral Assumption: We assume that each firm chooses its price p_i to maximize its profit, taking its rival's price (and the demand function for its own product) as given.

Firm 1's revenue and profit are given by

$$\pi_1(p_1, p_2) = (30 - \tfrac{2}{3}p_1 + \tfrac{1}{3}p_2)p_1 = (30 + \tfrac{1}{3}p_2)p_1 - \tfrac{2}{3}p_1^2.$$

The firm's first-order condition is

$$\frac{\partial \pi_1}{\partial p_1} = 0, \qquad i.e. \quad 30 + \tfrac{1}{3}p_2 - \tfrac{4}{3}p_1 = 0.$$

The firm's reaction function is therefore

$$p_1 = r_1(p_2) = 22\tfrac{1}{2} + \tfrac{1}{4}p_2.$$

The firms' demand functions are symmetric to one another, so Firm 2's reaction function is

$$p_2 = r_2(p_1) = 22\tfrac{1}{2} + \tfrac{1}{4}p_1.$$

Bertrand Equilibrium

If firms behave according to the Bertrand assumption, then the natural definition of equilibrium is a profile of prices in which each firm is taking the other firms' prices as given:

Definition: A **Bertrand equilibrium** in a market with n firms is a profile of prices, $(\widehat{p}_1, \ldots, \widehat{p}_n) \in \mathbb{R}^n_+$, that satisfies the condition

$$\forall i \in \{1, \ldots, n\} : \ \widehat{p}_i \text{ maximizes } \pi_i(p_i, \widehat{\mathbf{p}}_{-i}),$$

where $(p_i, \widehat{\mathbf{p}}_{-i})$ is the n-tuple $(\widehat{p}_1, \ldots, \widehat{p}_{i-1}, p_i, \widehat{p}_{i+1}, \ldots, \widehat{p}_n)$.

As with the Cournot equilibrium, a Bertrand equilibrium is therefore a simultaneous solution of the n firms' first-order equations. In our two-firm numerical example the first-order conditions are

$$30 + \tfrac{1}{3}p_2 - \tfrac{4}{3}p_1 = 0 \quad \text{and} \quad 30 - \tfrac{4}{3}p_2 + \tfrac{1}{3}p_1 = 0$$

and the unique solution is

$$p_1 = p_2 = \$30, \quad q_1 = q_2 = 20, \quad \text{and profits are } \pi_1 = \pi_2 = \$600.$$

Figure 6 depicts the two firms' reaction functions and the Bertrand equilibrium.

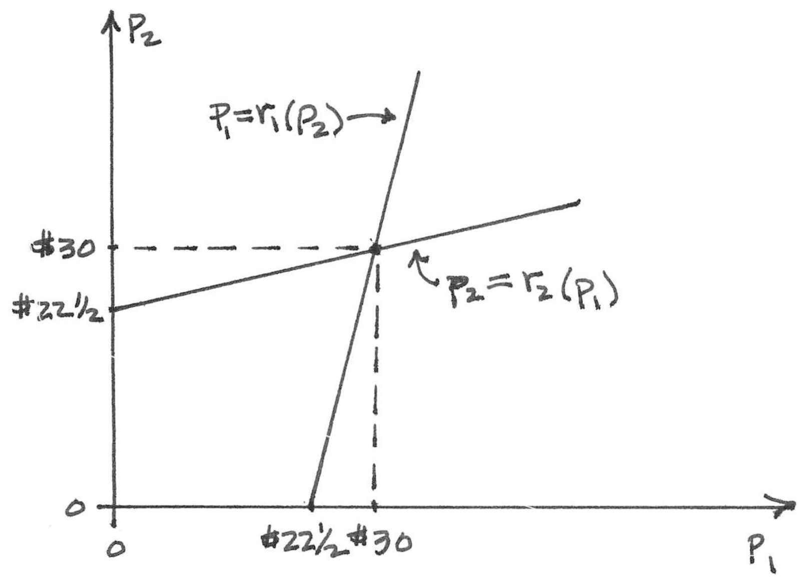

Figure 6

Comparison of the Bertrand and Cournot Equilibria

In the monopoly model we found that it makes no difference whether we regard the monopolist as choosing price or choosing quantity: the market demand curve dictates the quantity sold if the firm chooses the price, and the demand curve dictates the market-clearing price if the firm chooses the quantity it will bring to market. When we have multiple firms in the market let's see how the Bertrand "competing in prices" model and the Cournot "competing in quantities" model compare with one another.

In our numerical example, we expressed the demands for the firms' products in terms of the quantities they could sell, determined by the prices the two firms charge. Writing this in matrix form, we have

$$\mathbf{q} = \mathbf{b} - A\mathbf{p}, \qquad i.e., \quad \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

But we could equivalently express the demands in "inverse demand function" form:

$$\mathbf{p} = A^{-1}(\mathbf{b} - \mathbf{q}) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 30 - q_1 \\ 30 - q_2 \end{bmatrix} = \begin{bmatrix} 90 - 2q_1 - \ q_2 \\ 90 - \ q_1 - 2q_2 \end{bmatrix}$$

The firms' profit functions, expressed in terms of quantities, are

$$\widetilde{\pi}_1(q_1, q_2) = (90 - q_2)q_1 - 2q_1^2$$
$$\widetilde{\pi}_2(q_1, q_2) = (90 - q_1)q_2 - 2q_2^2,$$

and their first-order conditions for profit-maximazation are

$$90 - q_2 - 4q_1 = 0 \quad \text{and} \quad 90 - q_1 - 4q_2 = 0.$$

The Cournot equilibrium is therefore

$$q_1 = q_2 = 18, \quad p_1 = p_2 = \$36, \quad \text{and profits are } \widetilde{\pi}_1 = \widetilde{\pi}_2 = \$648.$$

This is a surprise! When there was just one firm, it made no difference whether the firm's decision variable was price or quantity. Here, when there are two firms, we get a different outcome depending on whether we model the firms as "competing in prices" or "competing in quantities."

Exercise: What is the explanation of this seeming paradox?

An Asymmetric Example

In the numerical example we used, the demand functions for the firms' products were symmetric and the firms supplied their products costlessly. The example was therefore a very special case, but it was used because the numbers and expressions were simple and transparent. Here's another example, in which the firms are not symmetric and their costs are not zero:

$$\begin{array}{lll} \text{Firm 1:} & q_1 = D_1(p_1, p_2) = 120 - 30p_1 + 20p_2 & C_1(q_1) = 4q_1 \\ \text{Firm 2:} & q_2 = D_2(p_1, p_2) = 240 + 10p_1 - 20p_2 & C_2(q_2) = 8q_2 \end{array}$$

You can derive the firms' reaction functions, which are

$$p_1 = r_1(p_2) = 4 + \tfrac{1}{3}p_2 \quad \text{and} \quad p_2 = r_2(p_1) = 10 + \tfrac{1}{4}p_1.$$

Figure 7 depicts the reaction functions and the Bertrand equilibrium, which is

$$p_1 = \$8,\ p_2 = \$12, \quad q_1 = 120,\ q_2 = 80, \quad \text{and profits are } \pi_1 = \$480,\ \pi_2 = \$320.$$

You can also invert the demand functions to obtain the inverse demand functions

$$p_1 = 18 - \tfrac{1}{20}q_1 - \tfrac{1}{20}q_2 \quad \text{and} \quad p_2 = 21 - \tfrac{1}{40}q_1 - \tfrac{3}{40}q_2,$$

which you can use to obtain the Cournot equilibrium (unfortunately, the numbers here are not nice ones):

$$q_1 = 105\tfrac{5}{11},\ q_2 = 69\tfrac{1}{11}, \quad p_1 \approx \$9.27,\ p_2 \approx \$13.18, \quad \text{and profits are } \pi_1 \approx \$556,\ \pi_2 \approx \$358.$$

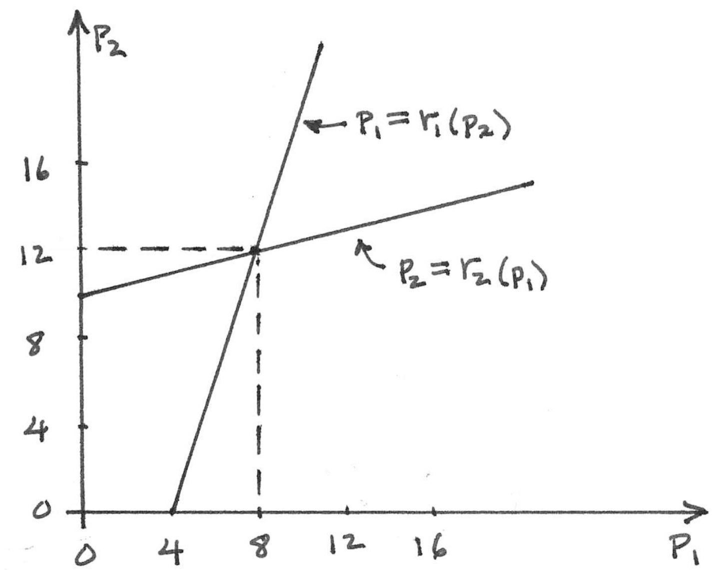

Figure 7