Consumer Surplus and Pareto Improvements
with Quasilinear Utility Functions

We've seen many examples in which a utility function has the *quasilinear* form $u(x, y) = y + v(x)$, such as

$$u(x, y) = y + 12x - (1/2)x^2 \quad \text{in Exercise \#3.5}$$

and $$u(x_0, x_H, x_F) = x_0 + 5\log x_H + 6\log x_F \quad \text{in Exercise \#9.5 ,}$$

and the two utility functions in the extended example in the "Equilibrium Under Uncertainty" section of the course.

In utility functions of this form utility is measured in the same units as the *y*-good: every unit increase in *y* is a unit increase in utility; and if an increase or decrease in *x* yields a change in utility of Δu, the same change in utility could have been brought about by instead changing *y* by the amount Δu.

Consumer surplus has a very nice interpretation for quasilinear utility functions: it's equal to the gain in utility. For example, a consumer with the first utility function above, from Exercise #3.5, has the demand function $x = 12 - p$ for the *x*-good if we fix the *y*-good's price at $1. Suppose the consumer has $y = 100$ and $x = 0$; then her utility is 100. If she then purchases $x = 8$ at a price of $p = \$4$, leaving her with $y = 68$, her utility will be

$$u = 68 + (12)(8) - (1/2)64 = 132,$$

a gain in utility of 32 units, or $32 worth of utility. You can easily check that her consumer surplus is also $32.

This fact makes quasilinear utility functions especially useful and instructive. For example, if everyone's utility function has this form then the Pareto efficient allocations are exactly the allocations that maximize total consumer surplus (in the absence of production). Similarly, Pareto improvements can be found by identifying allocations that increase total consumer surplus and then transferring some of the *y*-good (i.e., some of the "surplus" utility created) from the gainers to the losers.

There's another, absolutely fundamental way to find trades that will make someone better off, and thus to find Pareto-improving exchanges. Note that in Exercise #3.5 Ann's and Bill's MRS's at the no-trade allocation are 4 and 0. Thus Ann values an additional orange much more than Bill does. A trade in which Ann gets one of Bill's oranges and in which Bill gets, say, $2 from Ann will likely make them both better off: Ann gives up only $2 for the orange (less than her MRS of 4 – i.e., less than the value she placed on the additional orange) and Bill gets $2 for the orange he gave up (more than his MRS of 0 – i.e., more than the value he placed on the orange). In fact, the $2 Ann pays is less than her MRS *after* the transaction, and the $2 Bill receives is larger than his MRS *after* the transaction, so this trade will *definitely* make them both better off. You should check that after this proposed exchange both Ann's and Bill's utilities are indeed larger. You should also be able to make a further exchange that transfers one more orange to Ann and again increases both utilities.

Exercises: Consumer Surplus and Quasilinear Utility

1. Assume that a consumer's preference can be represented by a quasilinear utility function: $u(x,y) = y + v(x)$ for nonnegative real numbers *x* and *y*, where $v(\cdot)$ is a continuously differentiable function, increasing and strictly concave. Suppose the price of the *y*-good is $1 per unit and the price of the *x*-good is *p* dollars per unit. Show that the consumer surplus this consumer obtains by purchasing according to her demand function is equal to her gain in utility – i.e., it is equal to the difference between her utility *after* her purchase and her utility *before* her purchase.

2. Make up a counterexample to show that the change in consumer surplus and the change in utility are not always equal. (Clearly, your counterexample will have to use a preference that cannot be represented by a quasilinear utility function.)

Extension of Exercise #3.5:

Before the hurricane and with no trade, the utility levels are $164 for Ann and $132 for Bill. (The *y*-good here is "dollars", for which the price is $1 each, and we're using the quasilinear utility functions that yield the given MRS functions, as in this document's first paragraph. Therefore utility is measured in the same units as the *y*-good – as described above – so it's effectively measured in dollars.) Show that at the equilibrium the consumer surpluses are $2 each for Ann and Bill.

Show that after the hurricane, at the price ceiling of four dollars imposed by the price-gouging law, the consumer surpluses are $24 and $8 respectively. Ann is clearly worse off after the hurricane (she has a lower utility), so how can it be that her consumer surplus is $22 larger? Similarly the total surplus is larger after the hurricane ($32 instead of $4), even though there are fewer oranges than before; how can this be? You must understand the simple answer to these two questions; otherwise you don't understand the concept of consumer surplus.

If you found a Pareto improving trade in part (d), it should yield consumer surpluses greater that $24 for Ann and greater than $8 for Bill.

Extension of Exercise #5.13:

Determine Amy's, Beth's, and Carol's utilities when there is no trade. Use the MRS's at the no-trade allocation to find a Pareto improvement in which one orange is transferred from Amy to Beth and one orange is transferred from Amy to Carol. Can you find a strict Pareto improvement on the no-trade allocation (i.e, each person is made strictly better off) in which one orange is transferred from Amy to Carol and in which Beth gets no additional oranges?

Determine each person's utility at the Walrasian equilibrium and their consumer surpluses at the Walrasian equilibrium. You should find that each person's consumer surplus is the difference between her utility at the equilibrium allocation and her utility at the no-trade allocation. Check whether the equilibrium allocation is a Pareto improvement on the no-trade allocation.

In the Extended Example of "Equilibrium Under Uncertainty":

(The arithmetic here involves a lot of fractions or decimal numbers. You'll probably find it more instructive and easier to check your work if you do everything in fractions, never converting to decimals – for example, convert the fraction 225/40 perhaps to 45/8 or 5 5/8 but not to 5.625.)

Determine each person's consumer surplus at any interior Pareto optimal allocation by calculating their utilities at the no-trade allocation and at the Pareto allocations. You should find that they have utility levels 43 1/8 and 33 3/4 at the no-trade allocation and that "total utility" is 78 at every interior Pareto allocation, so total consumer surplus at the Pareto allocations is 78 – 76 7/8 = 1 1/8.

Determine their consumer surpluses at the Arrow-Debreu equilibrium. You should find that CS_A = 27/40 and that CS_B = 18/40.

Determine their consumer surpluses at the credit-market-only equilibrium. You should find that CS_A = 3/40 and CS_B = 2/40.

Here's a slightly more challenging question: can you find a Pareto improvement on the credit-market-only allocation or on the no-trade allocation that does not involve any redistribution of consumption today? In other words, a Pareto improvement in which each person consumes 15 units today. You should be able to use the four MRS's for consumption tomorrow – i.e., MRS_H^A, MRS_L^A, MRS_H^B, and MRS_L^B – to find a Pareto-improving trade involving only tomorrow's consumption.