BERTRAND EQUILIBRIUM: AN EXAMPLE

THE FIRMS PRODUCE "DIFFERENTIATED" PRODUCTS AND CHOOSE THEIR PRICES STRATEGICALLY.*

	DEMAND	COSTS	
FIRM #1:	$q_1 = 120 - 30p_1 + 20p_2$	$C_1(q_1) = 4q_1$	$MC_1 = 4$
FIRM #2:	$q_2 = 240 + 10p_1 - 20p_2$	$C_2(q_2) = 8q_2$	$MC_2 = 8$

$$\pi_1(p_1, p_2) = p_1 q_1 - 4q_1$$

$$= 120p_1 - 30p_1^2 + 20p_2 p_1 - 4(120 - 30p_1 + 20p_2)$$

$$= 120p_1 - 30p_1^2 + 20p_2 p_1 - 480 + 120p_1 - 80p_2$$

$$\frac{\partial \pi_1}{\partial p_1} = 120 - 60p_1 + 20p_2 + 120$$

$$= 240 - 60p_1 + 20p_2$$

$= 0$ AT FIRM 1's BEST p_1

i.e., $60p_1 - 20p_2 = 240$

i.e., $6p_1 - 2p_2 = 24$

$\boxed{p_1 = 4 + \frac{1}{3}p_2}$ ← FIRM 1's REACTION FUNCTION

* IF THEY PRODUCE IDENTICAL (HOMOGENEOUS) PRODUCTS, THE ANALYSIS IS COMPLETELY DIFFERENT, DUE TO DISCONTINUITIES IN DEMAND. THIS IS TREATED IN SEPARATE NOTES.

$$\pi_2(p_1, p_2) = p_2 q_2 - 8q_2$$

$$= 240p_2 + 10p_1p_2 - 20p_2^2 - 8(240 + 10p_1 - 20p_2)$$

$$= 240p_2 + 10p_1p_2 - 20p_2^2 - 1920 - 80p_1 + 160p_2$$

$$\frac{\partial \pi_2}{\partial p_2} = 240 + 10p_1 - 40p_2 + 160$$

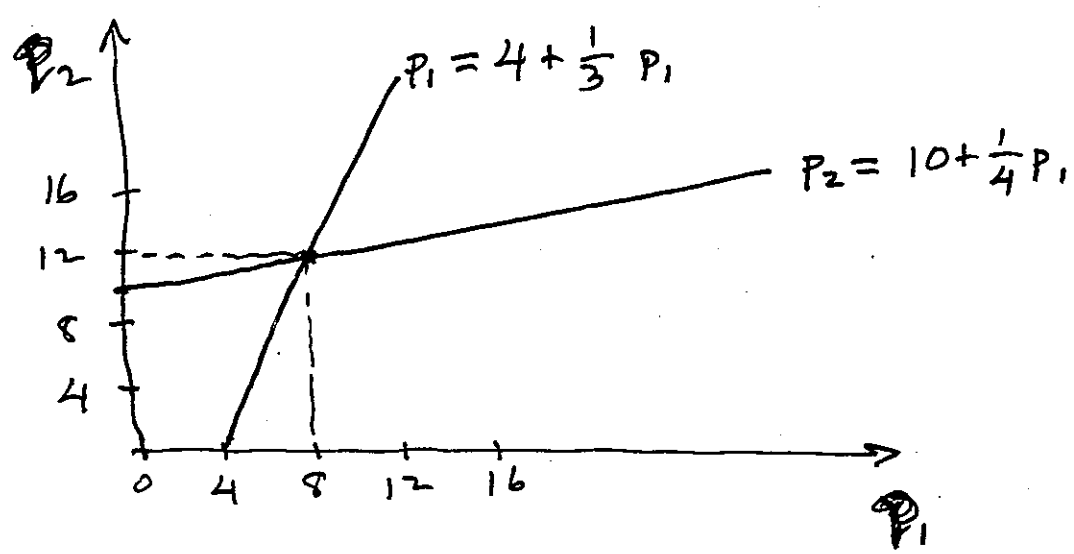
$$= 400 + 10p_1 - 40p_2$$

$= 0$ AT FIRM 2'S BEST p_2 ;

i.e., $40p_2 - 10p_1 = 400$

i.e., $4p_2 - p_1 = 40$

i.e., $\boxed{p_2 = 10 + \frac{1}{4}p_1}$ ← FIRM 2'S REACTION FUNCT



BERTRAND EQUILIBRIUM: $p_1 = \$8$, $p_2 = \$12$.

(OBTAINED BY SOLVING THE TWO PROFIT-MAXIMIZATION EQUATIONS — THE REACTION FUNCTIONS — SIMULTANEOUSLY)

THE BERTRAND EQUILIBRIUM:

$p_1 = \$8$ | $p_2 = \$12$

$q_1 = 120$ | $q_2 = 80$

$\pi_1 = (\$8 - \$4) \cdot 120$
$= \$480.$

$\pi_2 = (\$12 - \$8) \cdot 80$
$= \$320.$

WHAT WOULD THE FIRMS DO IF THEY COLLUDED?

TOTAL PROFIT (e.g. COLLUSION):

$$\pi(p_1, p_2) := \pi_1(p_1, p_2) + \pi_2(p_1, p_2)$$
$$= [-30p_1^2 + 240p_1 - 80p_2 + 20p_1p_2 - 480]$$
$$+ [-20p_2^2 + 400p_2 - 80p_1 + 10p_1p_2 - 1920]$$

$$\frac{\partial \pi}{\partial p_1} = \frac{\partial \pi_1}{\partial p_1} + \frac{\partial \pi_2}{\partial p_1} = [-60p_1 + 240 + 20p_2] + [-80 + 10p_2]$$
$$= -60p_1 + 30p_2 + 160$$

$$\frac{\partial \pi}{\partial p_2} = \frac{\partial \pi_1}{\partial p_2} + \frac{\partial \pi_2}{\partial p_2} = [-80 + 20p_1] + [-40p_2 + 400 + 10p_1]$$
$$= 30p_1 - 40p_2 + 320$$

AT $(p_1, p_2) = (8, 12)$:

$$\left\{ \begin{array}{l} \frac{\partial \pi}{\partial p_1} = [-480 + 240 + 240] + [-80 + 120] = 0 + 40 = 40 \\ \frac{\partial \pi}{\partial p_2} = [-80 + 160] + [-480 + 400 + 80] = 80 + 0 = 80 \end{array} \right.$$

$q_1 = 120 - 240 + 240 = 120$, $R_1 = 960$, $C_1 = 480$, $\pi_1 = 480$

$q_2 = 240 + 80 - 240 = 80$, $R_2 = 960$, $C_2 = 640$, $\pi_2 = 320$

$\pi = 800$

SHOULD INCREASE BOTH p_1, p_2.

SOLVING FOR COLLUSIVE (p_1, p_2):

$\frac{\partial \pi}{\partial p_1} = 0, \frac{\partial \pi}{\partial p_2} = 0$

$$60p_1 - 30p_2 = 160 \rightarrow 6p_1 - 3p_2 = 16$$
$$-30p_1 + 40p_2 = 320 \rightarrow -3p_1 + 4p_2 = 32$$
$$-6p_1 + 8p_2 = 64$$
$$5p_2 = 80; \quad p_2 = 16, \quad p_1 = \frac{64}{6} = \frac{32}{3} = 10\tfrac{2}{3}$$

CHECK:

$$6p_1 - 3p_2 = 64 - 48 = 16 \checkmark$$
$$-3p_1 + 4p_2 = -32 + 64 = 32 \checkmark$$

COLLUSION OUTCOME:

$$p_1 = \$10\tfrac{2}{3} = \frac{32}{3}, \quad p_2 = \$16$$
$$q_1 = 120 - (30)\left(\frac{32}{3}\right) + (20)(16) = 120 - 320 + 320 = 120$$
$$q_2 = 240 + (10)\left(\frac{32}{3}\right) - (20)(16) = 240 + \frac{320}{3} - 320 = \frac{80}{3} = 26\tfrac{2}{3}$$
$$R_1 = \left(\frac{32}{3}\right)(120) = 1280, \quad C_1 = 480, \quad \pi_1 = 800$$
$$R_2 = (16)\left(\frac{80}{3}\right) = \frac{1280}{3} = 426\tfrac{2}{3}, \quad C_2 = \frac{640}{3} = 213\tfrac{1}{3}, \quad \pi_2 = 213\tfrac{1}{3}$$
$$\pi = 1013\tfrac{1}{3}$$

CHECK FONC:

$$\frac{\partial \pi}{\partial p_1} = -60\left(\frac{32}{3}\right) + 30(16) + 160 = -640 + 480 + 160 = 0 \checkmark$$
$$\frac{\partial \pi}{\partial p_2} = 30\left(\frac{32}{3}\right) - 40(16) + 320 = 320 - 640 + 320 = 0 \checkmark$$

CONVERTING FROM BERTRAND TO COURNOT

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = A \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 120 \\ 240 \end{bmatrix}, \qquad A = \begin{bmatrix} -30 & 20 \\ 10 & -20 \end{bmatrix}$$

$$A \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \begin{bmatrix} 120 \\ 240 \end{bmatrix}$$

$$\therefore \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = A^{-1} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - A^{-1} \begin{bmatrix} 120 \\ 240 \end{bmatrix} ; \quad A^{-1} = \begin{bmatrix} -\frac{1}{20} & -\frac{1}{20} \\ -\frac{1}{40} & -\frac{3}{40} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{20} & -\frac{1}{20} \\ -\frac{1}{40} & -\frac{3}{40} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \begin{bmatrix} -\frac{1}{20} & -\frac{1}{20} \\ -\frac{1}{40} & -\frac{3}{40} \end{bmatrix} \begin{bmatrix} 120 \\ 240 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{20} q_1 - \frac{1}{20} q_2 + 6 + 12 \\ -\frac{1}{40} q_1 - \frac{3}{40} q_2 + 3 + 18 \end{bmatrix} = \begin{bmatrix} -\frac{1}{20} q_1 - \frac{1}{20} q_2 + 18 \\ -\frac{1}{40} q_1 - \frac{3}{40} q_2 + 21 \end{bmatrix}$$

$$\boxed{\begin{aligned} p_1 &= 18 - \tfrac{1}{20} q_1 - \tfrac{1}{20} q_2 \\ p_2 &= 21 - \tfrac{1}{40} q_1 - \tfrac{3}{40} q_2 \end{aligned}}$$

COURNOT EQUILIBRIUM:

$$\pi_1(q_1, q_2) = \left(18 - \tfrac{1}{20} q_2\right) q_1 - \tfrac{1}{20} q_1^2 - 4q_1$$

$$\frac{\partial \pi_1}{\partial q_1} = 18 - \tfrac{1}{20} q_2 - \tfrac{2}{20} q_1 - 4$$

$$= 14 - \tfrac{2}{20} q_1 - \tfrac{1}{20} q_2$$

$$= 0 \iff \tfrac{2}{20} q_1 + \tfrac{1}{20} q_2 = 14; \quad \text{i.e., } 2q_1 + q_2 = 280$$

$$\text{i.e., } q_1 = 140 - \tfrac{1}{2} q_2$$

$$\pi_2(q_1, q_2) = \left(21 - \tfrac{1}{40} q_1\right) q_2 - \tfrac{3}{40} q_2^2 - 8q_2$$

$$\frac{\partial \pi_2}{\partial q_2} = 21 - \tfrac{1}{40} q_1 - \tfrac{6}{40} q_2 - 8$$

$$= 13 - \tfrac{1}{40} q_1 - \tfrac{6}{40} q_2$$

$$= 0 \iff \tfrac{1}{40} q_1 + \tfrac{6}{40} q_2 = 13; \quad \text{i.e., } q_1 + 6q_2 = 520$$

$$\text{i.e., } q_2 = \tfrac{260}{3} - \tfrac{1}{6} q_1$$

SOLVING FOMC SIMULTANEOUSLY:

$$12q_1 + 6q_2 = 1680$$

$$q_1 + 6q_2 = 520$$

$$11q_1 = 1160$$

$$q_1 = \frac{1160}{11} = 105\,5/11$$

$$6q_2 = 520 - q_1$$

$$= \frac{5720}{11} - \frac{1160}{11} = \frac{4560}{11}$$

$$q_2 = \frac{760}{11} = 69\,1/11$$

$P_1 \cong 9.27$ $\quad P_2 \cong 13.18$

$R_1 \cong 977.9$ $\quad R_2 \cong 910.7$

$C_1 \cong 421.8$ $\quad C_2 \cong 552.7$

$\pi_1 \cong 556$ $\quad \pi_2 \cong 358$

$\pi \cong 914$

COURNOT REACTION CURVES & EQUILIBRIUM

REACTION FUNCTIONS:

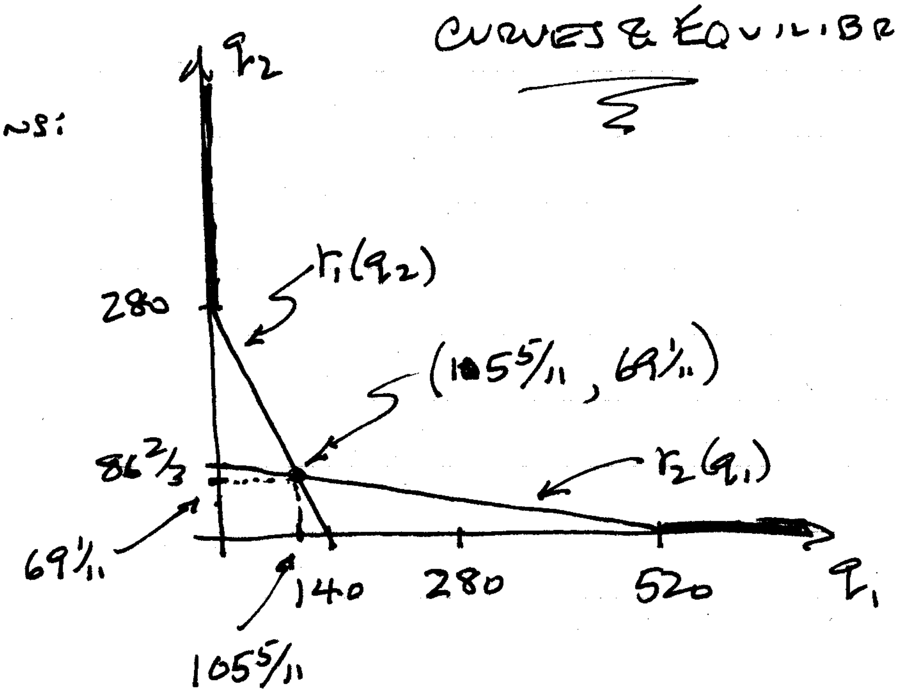
① $2q_1 + q_2 = 280$

② $q_1 + 6q_2 = 520$

i.e.,

$q_1 = 140 - \frac{1}{2}q_2$

$q_2 = \frac{260}{3} - \frac{1}{6}q_1$



NOTE THAT THIS OUTCOME IS DIFFERENT THAN THE BERTRAND ("FIRMS COMPETE IN PRICES") OUTCOME. THIS SHOULD SEEM SURPRISING. THE FIRM FACES A (RESIDUAL) DEMAND CURVE, AND IT SHOULDN'T MATTER WHETHER THE FIRM CHOOSES ITS PRICE (WITH THE DEMAND CURVE THEN DETERMINING ITS OUTPUT QUANTITY), OR THE FIRM CHOOSE ITS OUTPUT LEVEL (WITH THE DEMAND CURVE THEN DETERMINING ITS PRICE). WHAT'S THE EXPLANATION?

ON THE FOLLOWING PAGE WE OBTAIN THE COLLUSIVE OUTCOME USING QUANTITIES AS THE FIRMS' DECISION VARIABLES. THE COLLUSIVE OUTCOME IS THE SAME AS THE OUTCOME WE OBTAINED USING PRICES AS THE DECISION VARIABLES.

(WITH OUTPUTS q_1, q_2 AS THE DECISION VARIABLES)

COLLUSION:

$$\pi(q_1, q_2) = \left[\left(18 - \tfrac{1}{20}q_1 - \tfrac{1}{20}q_2\right)q_1 - 4q_1\right] + \left[\left(21 - \tfrac{1}{40}q_1 - \tfrac{3}{40}q_2\right)q_2 - 8q_2\right]$$

$$= \left[14q_1 - \tfrac{1}{20}q_2 q_1 - \tfrac{1}{20}q_1^2\right] + \left[13q_2 - \tfrac{1}{40}q_1 q_2 - \tfrac{3}{40}q_2^2\right]$$

$$\frac{\partial \pi}{\partial q_1} = \frac{\partial \pi_1}{\partial q_1} + \frac{\partial \pi_2}{\partial q_1} = \left[14 - \tfrac{1}{20}q_2 - \tfrac{2}{20}q_1\right] + \left[-\tfrac{1}{40}q_2\right]$$

$$= 14 - \tfrac{2}{20}q_1 - \tfrac{3}{40}q_2$$

$$= 0 \Leftrightarrow \tfrac{4}{40}q_1 + \tfrac{3}{40}q_2 = 14; \text{ i.e., } 4q_1 + 3q_2 = 560$$

$$\frac{\partial \pi}{\partial q_2} = \frac{\partial \pi_1}{\partial q_2} + \frac{\partial \pi_2}{\partial q_2} = \left[-\tfrac{1}{20}q_1\right] + \left[13 - \tfrac{1}{40}q_1 - \tfrac{6}{40}q_2\right]$$

$$= 13 - \tfrac{3}{40}q_1 - \tfrac{6}{40}q_2$$

$$= 0 \Leftrightarrow \tfrac{3}{40}q_1 + \tfrac{6}{40}q_2 = 13; \text{ i.e., } 3q_1 + 6q_2 = 520.$$

SOLVING SIMULTANEOUSLY:

$4q_1 + 3q_2 = 560$

$3q_1 + 6q_2 = 520$

$8q_1 + 6q_2 = 1120$

$5q_1 = 600$

$q_1 = 120,$

$\therefore 3q_2 = 560 - 480 = 80$

$q_2 = \frac{80}{3} = 26\tfrac{2}{3}$

$q_1 = 120 \qquad q_2 = \frac{80}{3} = 26\tfrac{2}{3}$

$P_1 = 18 - 6 - \frac{4}{3} = \$10\tfrac{2}{3}$

$P_2 = 21 - 3 - 2 = \$16$

THIS IS THE SAME COLLUSIVE OUTCOME WE OBTAINED WHEN WE ASSUMED P_1 AND P_2 WERE THE DECISION VARIABLES.

TWO REMARKS

(1) THE MONOPOLY ANALYSIS/MODEL IS ACTUALLY THE APPROPRIATE ANALYSIS FOR ANY FIRM FACING A DOWNWARD-SLOPING DEMAND CURVE WHICH IT TAKES AS GIVEN. DOWNWARD-SLOPING DEMAND CURVE = "MARKET POWER."

(2) IN THE COURNOT ("FIRMS COMPETE IN QUANTITIES") MODEL, THE FIRM FACES A DEMAND CURVE IT TAKES AS GIVEN, AND IT CHOOSES A q. BUT THAT IS EQUIVALENT TO CHOOSING A PRICE p — SO THE FIRM'S DECISION (p AND q) IS THE SAME EITHER WAY. AND THE SAME IN THE BERTRAND MODEL. BUT THE TWO MODELS' EQUILIBRIA DIFFER, FOR THE SAME FIRMS. HOW CAN THIS BE?